Factorizaciones en extensiones de ideales de monoides libres conmutativos

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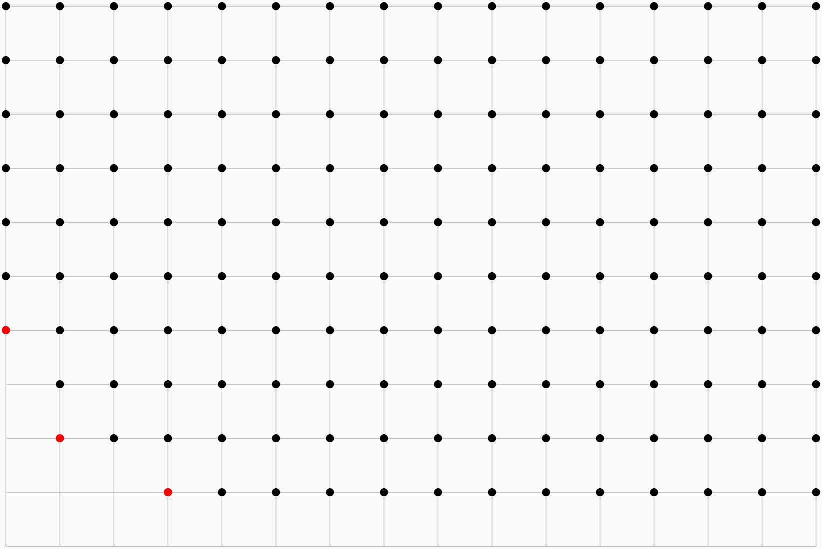
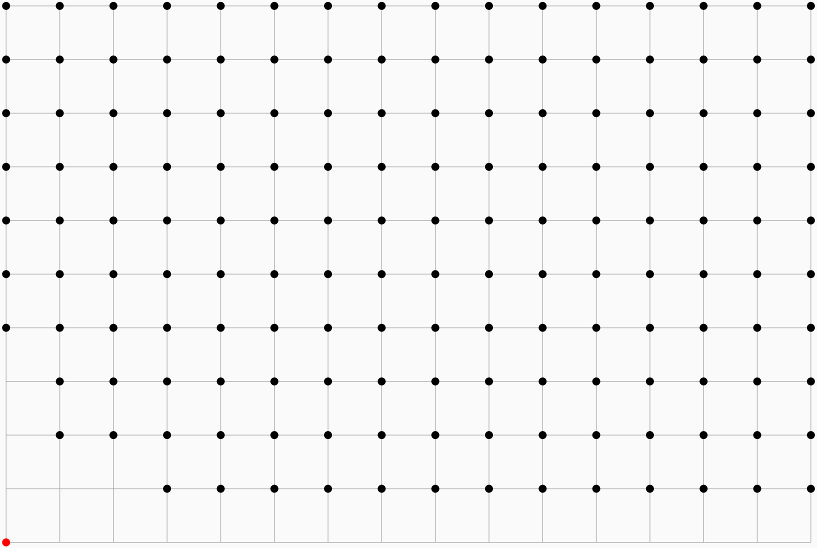
Ideals

$I = \{(0,4),(1,2),(3,1)\} + \mathbb{N}^2$; $I + \mathbb{N}^2 \subseteq I$



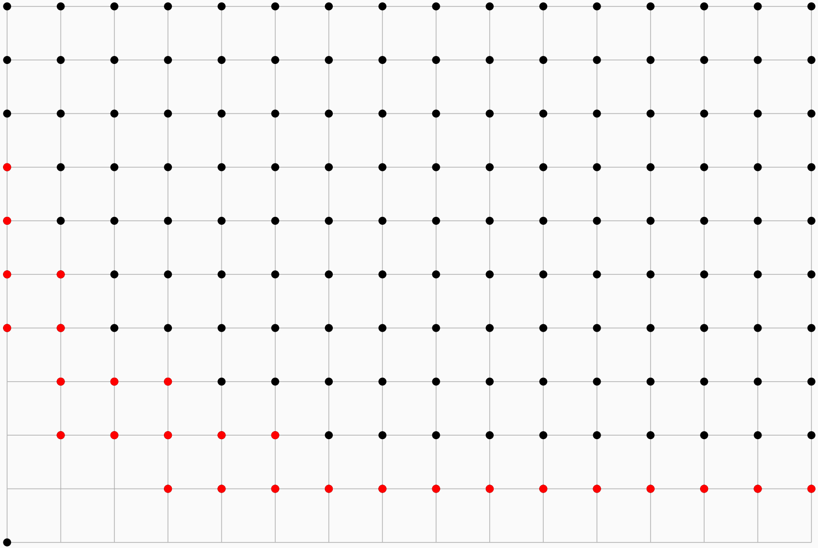
Ideal extensions

$S = \{0\} \cup I$; $S + S = S$; $S^* = I$



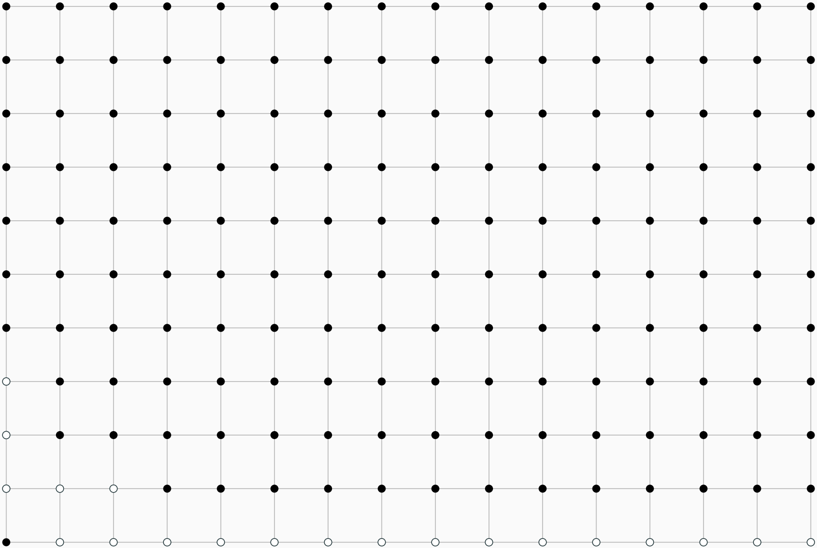
Atoms

$A = S^* \setminus (S^* + S^*)$



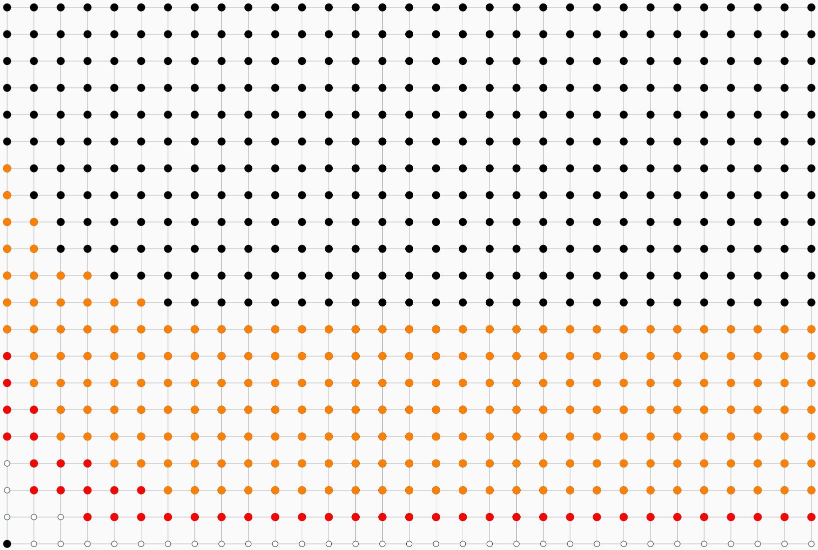
Gaps

$H = \mathbb{N}^2 \setminus S$



Gap absorbing monoid

(1) $H + H \subseteq H \cup A \cup (A + A)$ (2) $H + A \subseteq A \cup (A + A)$



Gap absorbing monoids and ideal extensions

Given $a, b \in \mathbb{N}^{(I)}$, write $a \leq b$ if $b - a \in \mathbb{N}^{(I)}$

$[\![a, b]\!] = \{x \in \mathbb{N}^{(I)} : a \leq x \leq b\}$

Let M be a submonoid of $\mathbb{N}^{(I)}$. The following are equivalent:

- M is a gap absorbing monoid.
- M is an ideal extension of $\mathbb{N}^{(I)}$ and for every $a, b \in A + A$, $[\![a, b]\!] \subseteq A + A$.

On $\mathbb{N}^2$, both concepts are equivalent

Conjecture: Every ideal extension is gap absorbing

Factorizations

Let S be an atomic monoid, $S = \langle A \rangle$

Let $F = \mathbb{N}^{(A)}$ be the free monoid on A

$$\varphi : F \to S, \varphi((\lambda_a)_{a \in A}) = \sum_{a \in A} \lambda_a a$$

is the *factorization morphism* of S

The set $\mathsf{Z}(s) = \varphi^{-1}(s)$ is the set of *factorizations* of s

Example: $(5,5)$ in our example has six different factorizations

$$\begin{aligned}(5,5) &= (2,3) + (3,2) = (2,2) + (3,3) \\ &= (1,4) + (4,1) = (1,3) + (4,2) \\ &= 2(1,2) + (3,1) = (0,4) + (5,1)\end{aligned}$$

Lengths of factorizations

Let S be an atomic monoid, $S = \langle A \rangle$

Let $z = (z_a)_{a \in A}$ be a factorization of $s \in S$. The *length* of z is

$$|z| = \sum_{a \in A} z_a$$

The set of lengths of factorizations of s is

$$\mathsf{L}(s) = \{|z| : z \in \mathsf{Z}(s)\}$$

Baeth's conjecture: If S is an ideal extension of $\mathbb{N}^{(I)}$ with finitely many gaps, then $\mathsf{L}(s)$ is an interval for every $s \in S$

Betti elemens (or degrees)

Given $s \in S$ with $S = \langle A \rangle$, we define the graph $\mathbf{G}_s$ as follows

- Vertices of $\mathbf{G}_s$: $\mathrm{Z}(s)$
- Edges of $\mathbf{G}_s$: zz' such that $z \cdot z' \neq 0$

The elements s of S such that $\mathbf{G}_s$ is not connected are called *Betti elements* of S; $\mathrm{Betti}(S)$ is the set of Betti elements of S

Example: $(5,5)$ is a Betti element in our example

Result: If S is gap absorbing with set of atoms A, then $\mathrm{Betti}(S) \subseteq (A+A) \cup (A+A+A)$

Conjecture: If S is an ideal extension of $\mathbb{N}^{(I)}$, then $\mathrm{Betti}(S) \subseteq A+A$

Delta sets

Let $s \in S$ and let $\mathsf{L}(s) = \{l_1 < \dots < l_r < \dots\}$

The Delta set of s is

$$\Delta(s) = \{l_r - l_{r-1} : r \geq 2\}$$

The monoid S is a BF-monoid if $\mathsf{L}(s)$ is finite for every $s \in S$

We know that on a BF-monoid, $\max\{\Delta(s) : s \in S\}$ is attained at a Betti element

Result: If S is gap absorbing, then $\mathsf{L}(s)$ is an interval for every $s \in S$

Catenary degree

Let z and z' be two factorizations of $s \in S$

The distance between z and z' is

$$\mathrm{d}(z, z') = \max\{|z - (z \wedge z')|, |z' - (z \wedge z')|\},$$

where $z \wedge z' = (\min\{z_a, z'_a\})_{a \in A}$

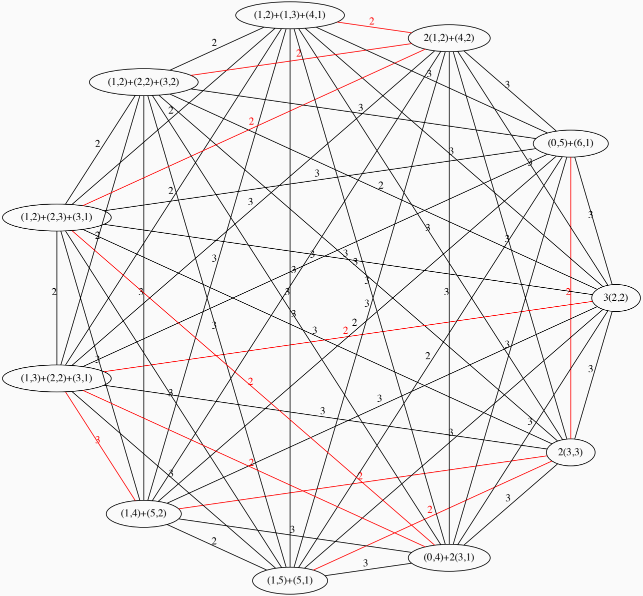
An N-chain of factorizations joining z and z' is a sequence of factorizations $z = z_0, z_1, \ldots, z_r = z'$ such that $\mathrm{d}(z_i, z_{i+1}) \leq N$

The catenary degree of s is the minimum N such that there exists an N-chain of factorizations joining any two factorizations of s

The catenary degree of S is the supremum of the catenary degrees of its elements, and it is attained at a Betti element

Result: If S is gap absorbing, then the catenary degree of S is at most four (conjectured to be at most three)

Example: $c((6,6)) = 3$



ω-primality

Given $a, b \in S$, we say that a divides b if $b - a \in S$, $a \leq_S b$

We say that a is a prime element of S if it is not a unit and whenever $a \leq_S b + c$, then $a \leq_S b$ or $a \leq_S c$

There are not primes in S unless $S = \mathbb{N}^{(I)}$

The ω-primality of $s \in S$, $\omega(s)$, is the least integer n such that whenever $s \leq_S (s_1 + \cdots + s_r)$, there exists $I \subseteq \{1, \ldots, r\}$ such that $|I| \leq n$ and $s \leq_S \sum_{i \in I} s_i$

The ω-primality of S is the supremum of the ω-primality of its atoms; denoted $\omega(S)$

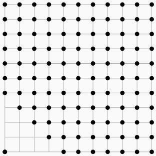
Result: If S is an ideal extension of $\mathbb{N}^{(I)}$ and a is an atom of S, then $\omega(a) \leq 1 + \|a\|_1$; if there exists $b \in S^*$ with $a \wedge b = 0$, then $\|a\|_1 \leq \omega(a)$

Notable examples

For S an ideal extension of $\mathbb{N}^2$ we know that it is gap absorbing, Betti$(S) \subseteq A + A$, and c$(S) \leq 3$

Let I be a set of non-negative integers and $\emptyset \neq J \subseteq I$, and let T be a numerical semigroup. The backslash monoid associated to I, J, and T is

$$S_I^J(T) = \left\{ x \in \mathbb{N}^{(I)} : \sum_{j \in J} x_j \in T \right\}$$



Result: $S_I^J(T)$ is gap absorbing if and only if $T = \{0\} \cup (m + \mathbb{N})$ (an ideal extension of $\mathbb{N}$; ordinary numerical semigroup)

In this setting, if $|I| > 1$, Betti$(S_I^J(T)) \subseteq A + A$, c$(S_I^J(T)) = 3$, $\omega(S_I^I(T)) = 2m - 1$, and $\omega(S_I^J(T)) = \infty$ for $J \subsetneq I$

Thank you for your attention!

More information at https://arxiv.org/abs/2311.06901

Check for other activities of the DAM network at https://dam-network.github.io/activities/