Towards a realization of the k-associahedron?

Multitriangulations and rigidity

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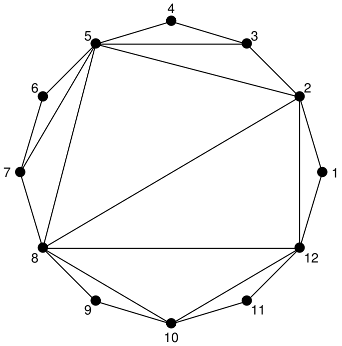
Multitriangulations

Triangulations

Consider n points in convex position in the plane, labeled $\{1, \ldots, n\}$ in cyclical order.
A triangulation of the n-gon is a maximal straightline graph on them with no crossings.



Triangulations

Many nice properties:

- All triangulations have the same number of edges ($2n - 3$) and triangles ($n - 2$).
- They are counted by Catalan numbers.
- They can all be constructed iteratively adding “ears” to a triangle.
- They can be connected by flips, forming (the graph of) a polytope (the associahedron).

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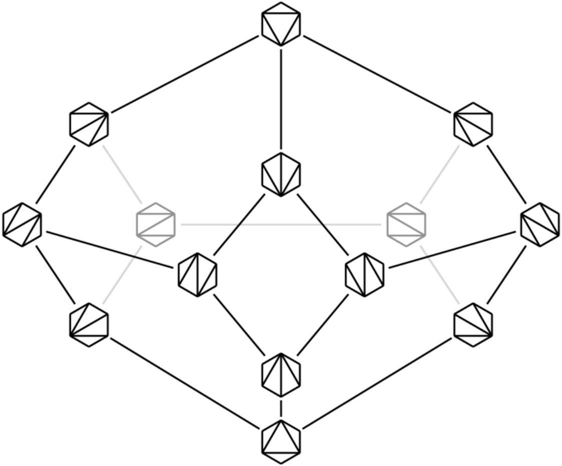
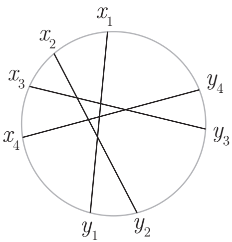


The 3-associahedron

k-crossings

Definition

A *k-crossing* is a set of k edges in $\binom{[n]}{2}$ that mutually cross.



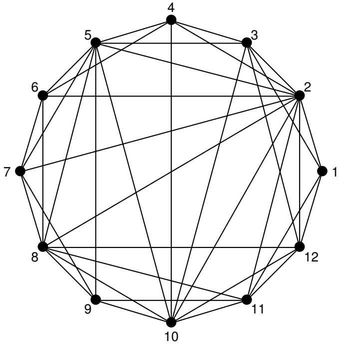
A 4-CROSSING

Remark: The definition is purely combinatorial. A k-crossing is a set $\{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\} \subset \binom{[n]}{2}$ of k edges with

$$i_1 < i_2 < \cdots i_k < j_1 < \cdots < j_k < i_1 \text{ (cyclically)}.$$

k-triangulations

A *k-triangulation* is a maximal graph with no $(k+1)$-crossings.

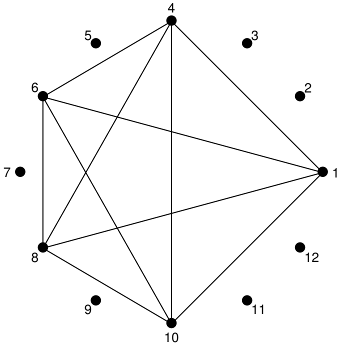


A 2-TRIANGULATION OF THE 12-GON

k-triangulations

Two easy constructions

- Start with a complete graph on $2k+1$ of the points and iteratively add "k-ears"



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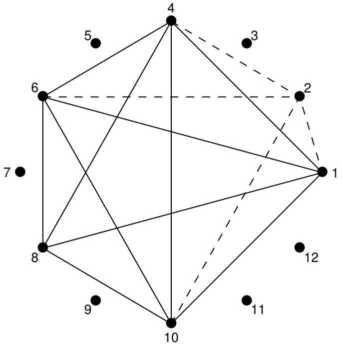

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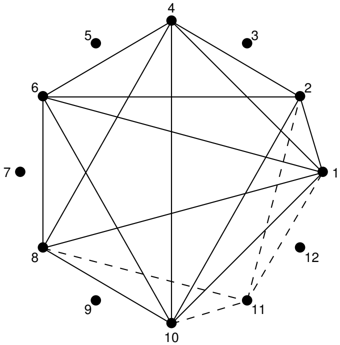

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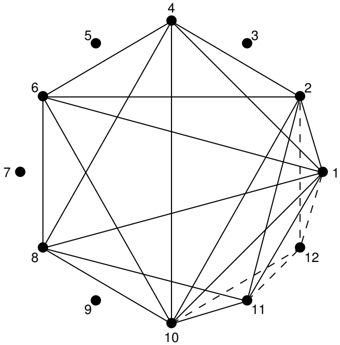

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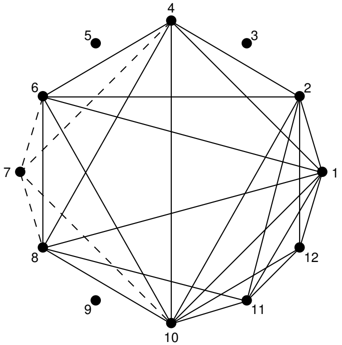

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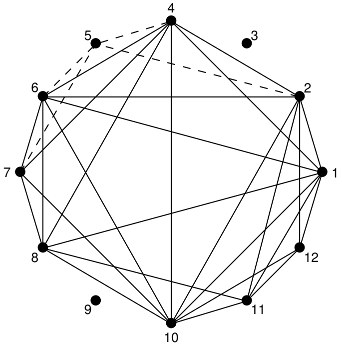

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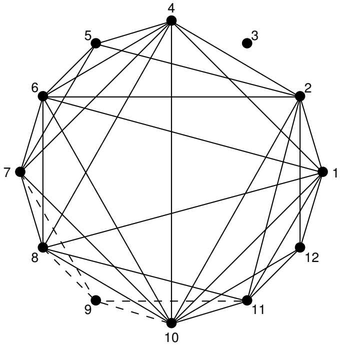

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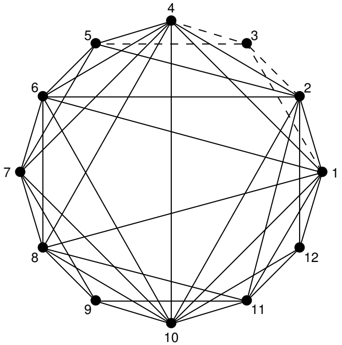

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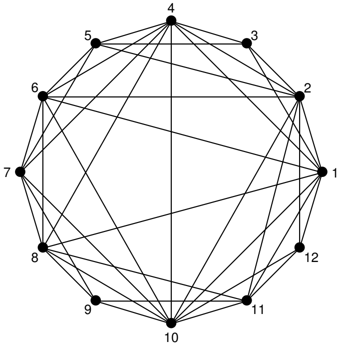
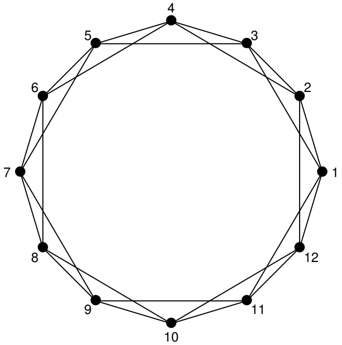


k-triangulations

Two easy constructions

- Start with the edges of length at most k and add all edges from nodes $1, \ldots, k$.



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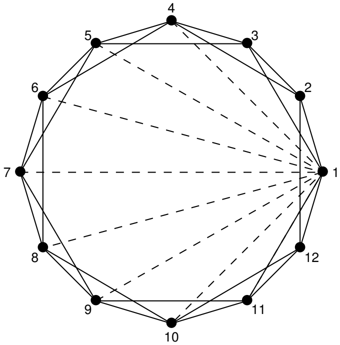

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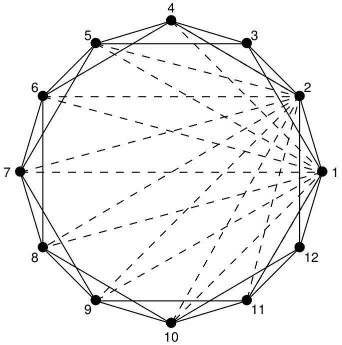

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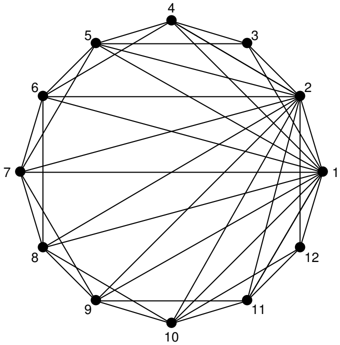

k-triangulations

Theorem (Capoyleas-Pach 1992, Nakamigawa 2000, Dress-Moulton-Koolen 2002)

All k-triangulations of the n-gon have the same number of edges, equal to $2kn - \binom{2k+1}{2}$.
Moreover, they are connected by "flips" (operations that remove an edge ans insert another).

k-associahedron

Is there a "polytope of k-triangulations" of the n-gon?

The associahedron as a simplicial complex

Asso(n) = the simplicial complex with vertices the $\binom{n}{2}$ diagonals of the n-gon and having as faces the the crossing-free sets of diagonals.
= clique complex of the crossing relation among the $\binom{n}{2}$ diagonals.

Vertices $= \binom{[n]}{2} = \{\{i, j\} : 1 \leq i < j \leq n\}$

Maximal faces ("facets") = triangulations of the n-gon.

Minimal non-faces = crossings.

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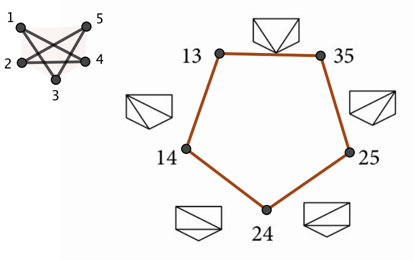
The associahedron as a simplicial complex

Example: Asso(5).



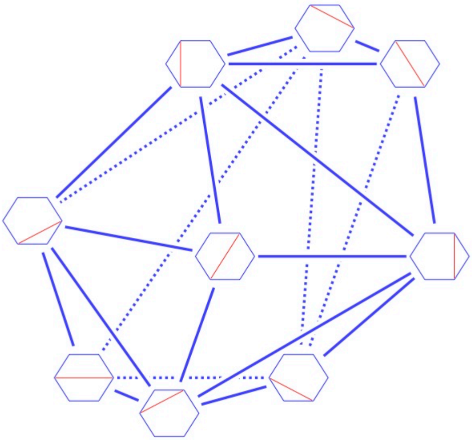
Remark: the "irrelevant edges" $\{i, i+1\}$ are not shown in the complex. Formally, we distinguish between Asso(n), with $\binom{n}{2}$ vertices and dimension $2n-4$, and $\overline{Asso}(n)$, with $\binom{n}{2}-n$ vertices and dimension $n-4$.

Theorem (Tamari-Stasheff-Milnor-Haiman, Lee 1989)

$\overline{Asso}(n)$ is a polytopal $(n-4)$-sphere. That is, there is a simplicial $(n-3)$-polytope with face poset isomorphic to it.

The 3-dimensional (simplicial) associahedron

$n = 6$: $\overline{Asso}(6)$ is a 2-sphere with 9-vertices, 21 edges, and 14 triangles.



The k-associahedron

DEFINITION: $\mathrm{Asso}_k(n)$ = the simplicial complex with vertices the $\binom{n}{2}$ diagonals of the n-gon and whose faces are the sets of diagonals containing no $(k+1)$-crossing.

$\overline{Asso}_k(n)$ =the subcomplex induced by the relevant edges (edges of length greater than k).

Maximal faces = k-triangulations of the n-gon.
Minimal non-faces = $(k+1)$-crossings.

Theorem (Jonsson 2003)

$\overline{Asso}_k(n)$ is a shellable sphere of dimension $k(n-2k-1)-1$

The main conjecture

Conjecture 1 (Folklore?, Jonsson?)

The shellable sphere $\overline{\text{Asso}}_k(n)$ is polytopal.

That is, there is a simplicial polytope of dimension $k(n-2k-1)$ with face poset isomorphic to the inclusion poset of subsets of diagonals of the n-gon not containing a $k+1$-crossing.

- True for $n \leq 2k+3$.
- True for $(k, n) = (2, 8)$ (Bokowski and Pilaud, 2009)
- True for $(2, 9), (2, 10), (3, 10)$ (Crespo-S. 2023+, this talk).

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A weaker conjecture

Conjecture 1'

The shellable sphere $\overline{\text{Asso}}_k(n)$ is geodesic (a.k.a. star-convex).

That is, there is a **complete simplicial fan** of dimension $k(n-2k-1)$ with face poset isomorphic to the inclusion poset of subsets of diagonals of the n-gon not containing a $k+1$-crossing.

The weaker conjecture holds for

- $n \leq 2k+4$ (Bergeron-Ceballos-Labbé, 2015)
- $k=2$ and $n \leq 13$ (Manneville 2017).
- $(3,11)$ and $(4,13)$ (Crespo-S. 2023+, this talk).

This includes every (k,n) with $n \leq 13$ except $(3,12)$ and $(3,13)$

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Rigidity

Bar-and-joint (infinitesimal) rigidity

Let $\mathbf{p} = \{p_1, \dots, p_n\} \in \mathbb{R}^d$ be points and let $G = ([n], E)$ be a graph. We call the pair $(G, \mathbf{p})$ a framework.

The framework is (infinitesimally) flexible if there is a non-trivial assignment of velocities $v_1, \dots, v_n \in \mathbb{R}^d$ to the points that maintains all distances in the graph. That is,

$$\langle v_i - v_j, p_i - p_j \rangle = 0 \quad \text{for every } \{i, j\} \in E.$$

If this does not happen, we say $(G, \mathbf{p})$ is (infinitesimally) rigid.

Theorem (Maxwell?)

Suppose that $\mathbf{p}$ *affinely spans* $\mathbb{R}^d$*. Then rigid frameworks on* $\mathbf{p}$ *are the spanning sets of rows of a matrix of size* $\binom{n}{2} \times nd$ *and rank* $nd - \binom{d+1}{2}$*.*

The rigidity matrix

$$R(\mathbf{p}) := \begin{pmatrix} p_1 - p_2 & p_2 - p_1 & 0 & \dots & 0 & 0 \\ p_1 - p_3 & 0 & p_3 - p_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ p_1 - p_n & 0 & 0 & \dots & 0 & p_n - p_1 \\ 0 & p_2 - p_3 & p_3 - p_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p_{n-1} - p_n & p_n - p_{n-1} \end{pmatrix}.$$

This in particular defines the rigidity matroid $\mathcal{R}(\mathbf{p})$ of $\mathbf{p}$, with $\binom{n}{2}$ elements and rank $nd - \binom{d+1}{2}$.

A numerical coincidence

If we let $d = 2k$ then the rank of the rigidity matrix equals

$$2nk - \binom{2k+1}{2} = \text{size of every } k\text{-triangulation.}$$

This led us to conjecture

Conjecture 2 (Pilaud-S. 2009)

k-triangulations are bases in the rigidity matroid for some (hence for any generic) choice of points $\mathbf{p} \subset \mathbb{R}^{2k}$.

Relation btw. Conjectures 1 and 2

If k-triangulations are rigidity bases (Conjecture 2) then the rows of $R(\mathbf{p})$ (for a valid $\mathbf{p}$) provide a vector configuration in which every k-triangulation spans a simplicial cone of the right dimension.

This configuration might be the set of normal vectors of a simplicial fan realizing $\overline{\mathrm{Asso}_k(n)}$ ($\Rightarrow$ Conjecture 1').

Hopefully, the fan is polytopal ($\Rightarrow$ Conjecture 1).

Status of Conjecture 2

- It holds for $k = 2$ (Pilaud-S., 2009).
- In all cases where Conjecture 1 is known, Conjecture 2 is known too.
- For every $k \geq 3$ and $n \geq 2k + 3$ there is a $\mathbf{p}$ along the moment curve that is not valid: it makes some k-triangulation dependent. (Crespo-S. 2023+).

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Two alternative forms of rigidity

As before, let $\mathbf{p} = \{p_1, \ldots, p_n\} \in \mathbb{R}^d$ be points, and consider the following modified rigidity matrix:

$$H(\mathbf{p}) := \begin{pmatrix} p_2 & -p_1 & 0 & \ldots & 0 & 0 \\ p_3 & 0 & -p_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ p_n & 0 & 0 & \ldots & 0 & -p_1 \\ 0 & p_3 & -p_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & p_n & -p_{n-1} \end{pmatrix}.$$

Kalai's hyperconnectivity matrix / matroid

Two alternative forms of rigidity

Let now $\mathbf{q} = \{(x_1, y_1), \dots, (x_n, y_n)\} \in \mathbb{R}^2$ be points, choose a "degree" $d \in \mathbb{N}$, and consider the following modified rigidity matrix:

$$C_d(\mathbf{q}) := \begin{pmatrix} c_{1,2} & -c_{1,2} & 0 & \dots & 0 & 0 \\ c_{1,3} & 0 & -c_{1,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_{n,2} & 0 & 0 & \dots & 0 & -c_{1,n} \\ 0 & c_{2,3} & -c_{2,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c_{n-1,n} & -c_{n-1,n} \end{pmatrix},$$

with $c_{ij} := \left(x_{ij}^{d-1}, y_{ij}x_{ij}^{d-2}, \dots, y_{ij}^{d-1}\right), \quad x_{ij} = x_i - x_j, \quad y_{ij} = y_i - y_j.$

Whiteley's cofactor matrix / matroid

Two alternative forms of rigidity

Theorem (Kalai 1985, Whitely 1990)

For $\mathbf{p}$ or $\mathbf{q}$ in general position, the (row vectors of) matrices $H(\mathbf{p})$ and $C_d(\mathbf{q})$ share the following properties with $R(\mathbf{p})$:

1. *Their rank equals $nd - \binom{d+1}{2}$.*
2. *Every K_{d+2} is a circuit.*

Matroids in $\binom{[n]}{2}$ with these properties are precisely the abstract rigidity matroids of Graver 1991 (as proved by Nguyen 2010).

An important common case; the moment curve

Let $t = (t_1, \ldots, t_n)$ be real parameters, and consider the configurations $\mathbf{p}(t) \subset \mathbb{R}^d$ with $p_i = (t_1, \ldots, t_i^d)$ along the moment curve and $\mathbf{q}(t) \subset \mathbb{R}^2$ with $q_i = (t_1, t_i^2)$ along the parabola. Then

Theorem (Crespo-Santos 2023)

The matrices $R(\mathbf{p}(t))$, $H(\mathbf{p}(t))$ and $C_d(\mathbf{q}(t))$ are equivalent under multiplication on the left by a nonsingular matrix. In particular, the associated oriented matroids coincide.

We denote this common (oriented) matroid $\mathcal{P}_d(t)$, and call $\mathcal{P}_d(n)$ the generic one.

Conjecture 2' (Stronger than Conjecture 2)

k-triangulations of the n-gon are bases in $\mathcal{P}_{2k}(n)$.

Status: same as Conjecture 2 (Crespo-S. 2023+).

A bright idea

Cofactor rigidity of degree d shares most of the properties of bar-and-joint rigidity in dimension d, yet it is about points in the plane.

Maybe this is the right tool to embed the multiassociahedron.

Conjecture 3 (S., $\simeq$ 2021)

For every choice of points $\mathbf{q} = \{q_1, \dots, q_n\}$ in convex position, the rows of $C_{2k}(\mathbf{q})$ embed $\overline{\text{Asso}}_k(n)$ as a polytopal fan.

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Status of Conjecture 3:

- True for $k = 1$ (Rote-S.-Streinu 2003)
- FALSE for $k = 3, n \geq 9$ (Crespo-S., 2023+)

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Conjecture 3'

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Status of Conjecture 3':

- True for $k = 1$ (Rote-S.-Streinu 2003)
- FALSE for $k = 3, n \geq 12$ (Crespo-S., 2023+)

Multitriangulations and Rigidity

Polytopality via vector configurations

Our heuristics for politopality: Given a simplicial $(d-1)$-sphere Δ with vertex set $[n]$ and a vector configuration $V = \{v_1, \dots, v_n\} \subset \mathbb{R}^d$ we check three things (each stronger than the previous one):

1. Are all faces of Δ linearly independent in V? (compute ranks)
2. Is Δ a "triangulation of V" (a.k.a. simplicial fan)? (compute orientations)
3. Is Δ a "regular triangulation of V" (a.k.a. projective fan; a.k.a. the normal fan of a simplicial polytope)? (linear feasibility)

If successful, these three computations answer Conjectures 2, 1' and 1 in the positive, respectively.

Our experiments

We have implemented this with $\Delta = \overline{\mathrm{Asso}}_k(n)$ and with $V =$"rows of the cofactor matrix of n points along the parabola" (equivalently, "bar-and-joint with points along the moment curve").

There are two "natural" choices of points:

- Equispaced along the parabola: $t_i = i$, that is, $q_i = (i, i^2)$
- Equispaced along the circle: Vertices of a regular n-gon, sent to the parabola via projective transformation

(Remark: projective transformations preserve the three forms of rigidity).

Our experiments; $k = 2$

- With $k = 2$ all positions we have tried realize the complete fan, but not always the polytope. (We have been able to compute up to $n = 13$).
- Equispaced positions along the parabola give a polytopal fan for $n \leq 9$.
- Positions $t = (2, 1, 2, 3, 4, 5, 6, 7, 9, 20)$ give a polytopal fan for $n = 10$.
- We have not found positions giving a polytopal fan for $n > 10$ (but our experiments are not conclusive).

Conjecture 3" (S.-Crespo 2023)

For $k = 2$ and any n, all positions along the parabola / moment curve realize $\overline{Asso_2}(n)$ as a complete simplicial fan.

Our experiments; $k > 2$

- With $k = 3$ and $n \geq 9$ there are positions where some k-triangulations are not bases.
- With $k = 3$ and $n \leq 11$ (and $k = 4$ and $n \leq 13$) equispaced positions on the circle realize the fan.
- With $k = 3$ and $n \leq 10$ the positions $t = (2, 1, 2, 3, 4, 5, 6, 7, 9, 20)$ realize the polytope.
- With $k = 3$ and $n \geq 12$ (and $k > 3$ and $n \geq 2k + 6$) no positions realize the fan.

An obstruction

The last point is not an experiment, but a theorem:

Theorem (Crespo-S. 2023)

For any choice $\mathbf{q} = \{q_1, \ldots, q_{12}\} \subset \mathbb{R}^2$ *of points in convex position there is a* 3*-triangulation that does not get the right orientation as a cone in the row-vectors of cofactor rigidity* $C_3(\mathbf{q})$*.*

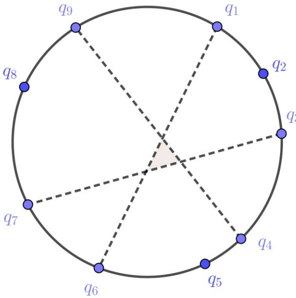
Idea of proof

- Let $T_9 := K_9 \setminus \{ 16, 37, 49 \}$. It is a 3-triangulation, and is also a triple cone over the graph of an octahedron.
- The graph of an octahedron is a circuit or a basis or in $C_3(6)$ depending on whether the three missing edges are concurrent or not (“Morgan-Scott obstruction”, 1975).
- When they are not concurrent, their sign as a rigidity basis is determined by the orientation of the triangle they form.
- Rigidity (both cofactor and bar-and-joint) behaves well with coning.

Idea of proof

Corollary

T_9 gets the correct orientation on nine given points if, and only if, the "inner half-planes" defined by the three missing edges $16, 37$, and 49 have non-empty intersection.



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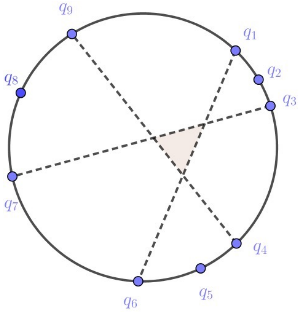

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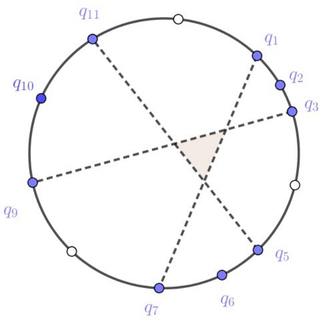
Idea of proof

Corollary

For any 12 points $\mathbf{q} = \{q_1, \ldots, q_{12}\} \subset \mathbb{R}^2$ in convex position either the 3 triangulation containing T_9 on $\mathbf{q} \setminus \{q_2, q_6, q_{10}\}$ or the one on on $\mathbf{q} \setminus \{q_4, q_8, q_{12}\}$ gets the wrong orientation.



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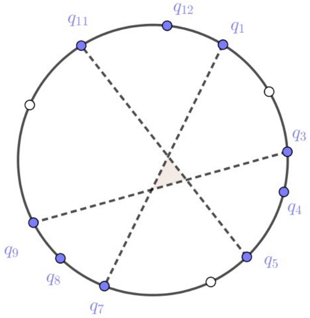

bad

Summing up

- Rigidity seemed a bright idea to realize the multiassociahedron. . . but it is proven not to work.
- Maybe the polytopality conjecture is false . . . This would be the first (?) family of "naturally defined" shellable simplicial spheres that turn out not to be polytopal.
- The case $k = 2$ of the polytopality conjecture may still be true.

A computational challenge

Is $\overline{\mathrm{Asso}}_3(12)$ polytopal? (42 vertices, 379 236 facets, dimension 14)

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The end

Thank you