Infinite Chains in the Tree of Numerical Semigroups

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Sesión 4: Matemática Discreta y Algorítmica

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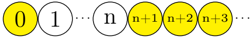
Outline

Numerical semigroups

A set $\Lambda \subset \mathbb{N}_0$ is a numerical semigroup if

it contains 0   it is closed for addition   it has a finite complement in $\mathbb{N}_0$

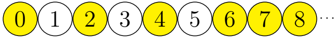
Examples

O_n : 

Ordinary semigroup

$G(O_n) = \{1, \dots, n\}$, the set of *gaps* of O_n

$g(O_n) = n$, the *genus* of O_n

H_3 : 

Hyperelliptic semigroup

$G(H_3) = \{1, 3, 5\}$

$g(H_3) = 3$

A problem. Is the sequence n_g formed by the number of numerical semigroups of genus g increasing? How does it grow?

- (2008 - M. Bras-Amorós) conjectured that n_g grows Fibonacci-like.

- (2012 - A. Zhai) proved $\frac{n_g}{\varphi^g} \xrightarrow{g\to\infty} S$, where φ is the golden ratio and S is at least 3.78.

It is not yet proven whether $\boxed{n_{g-2} + n_{g-1} \le n_g}$ or even $n_{g-1} \le n_g$.

Some invariants and notation

H_3 : 0 1 2 3 4 5 6 7 8 $\cdots$ *hyperelliptic semigroup of genus 3*

H_3 : λ_0 ◯ λ_1 ◯ λ_2 ◯ λ_3 λ_4 λ_5 $\cdots$ *enumeration Λ of H_3*

$m(H_3) = 2$, the *multiplicity* of H_3

$f(H_3) = 5$, the *Frobenius number* of H_3

$c(H_3) = 6$, the *conductor* of H_3

$\mathcal{L}(H_3) = \{0, 2, 4\}$, the *left elements* of H_3

$\langle 2, 7\rangle = H_3$, we say $2, 7$ are the *minimal generators* of H_3

The minimal generators that are not in $\mathcal{L}(\Lambda)$ are the right generators of Λ.

7 is the only right generator of H_3

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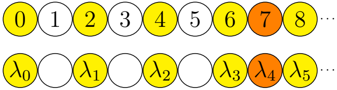

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The tree

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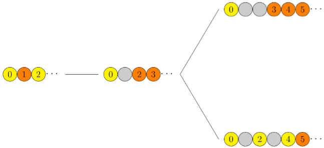

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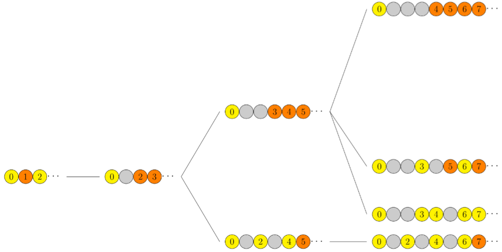

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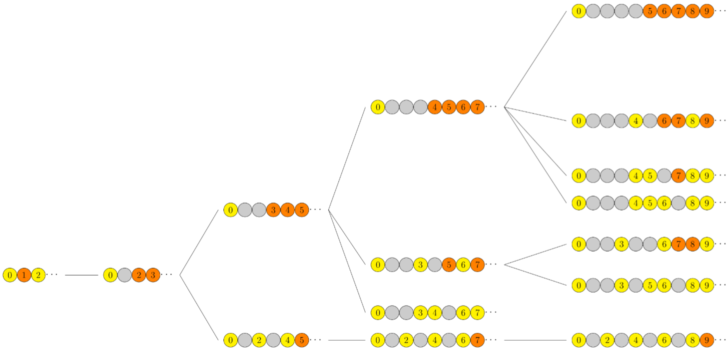

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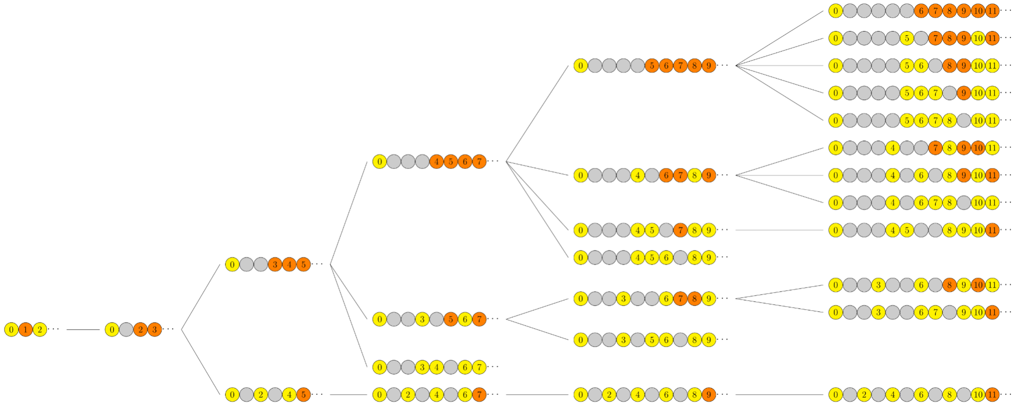
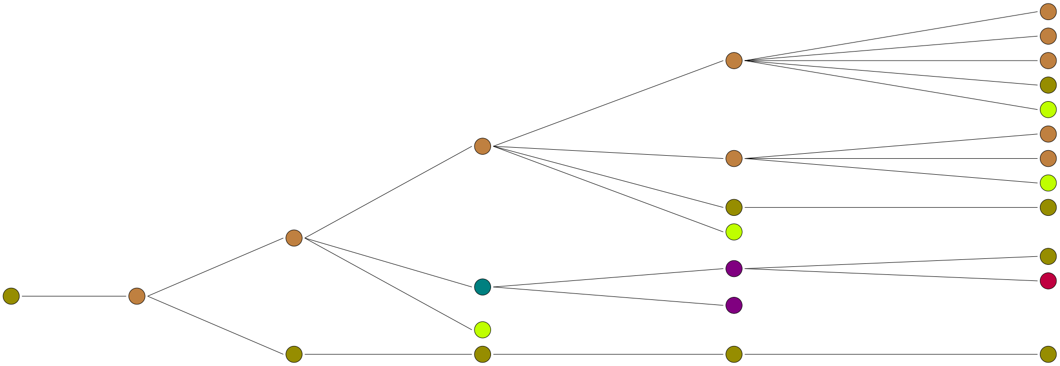


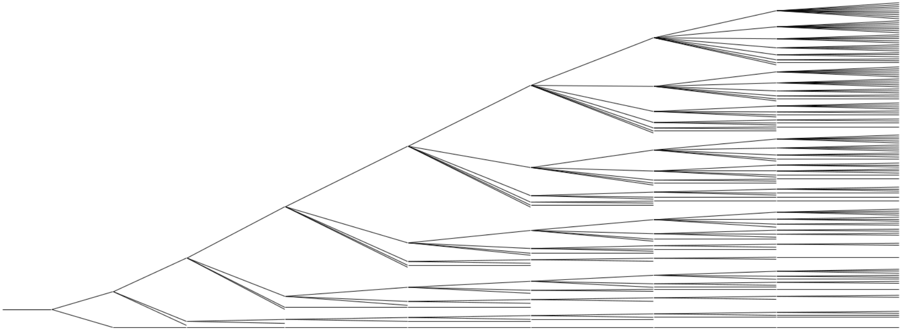
In tree,

Nodes are numerical semigroups

We have Child, Parent, Descendant

And Leaf, Stick, Bush





Tree structure of numerical semigroups up to level 9

Infinite chains

Lemma 1. Given an infinite chain $I = (\Lambda_i)_{i\geq 0}$ different than $I_{\mathcal{O}}$, it holds that $\bigcap_{i\geq 0} \Lambda_i = d \cdot \Lambda$ for some integer $d > 1$ and some numerical semigroup Λ.

Lemma 2. Given an integer $d > 1$ and a numerical semigroup Λ, the infinite chain obtained by deleting repetitions in the sequence $\Lambda_j = d \cdot \Lambda \cup \{l \in \mathbb{N} : l \geq j\}$ has intersection $d \cdot \Lambda$.

Consequently, $\mathbb{I} \setminus I_{\mathcal{O}}$ and $\mathbb{N}_{\geq 2} \times \mathbb{S}$ are in a one-to-one correspondence.

For example, in this correspondence, the image of $I_{\mathcal{H}} = \{\mathcal{H}_g : g \geq 0\}$ would be $(2, \mathbb{N}_0)$.

Definition. A descendant of a numerical semigroup *beyond* a given nongap of the semigroup is a descendant that contains all nongaps up to the given nongap.

Theorem. Let Λ be a non-ordinary numerical semigroup with enumeration λ, genus g, and conductor c, and let d be the greatest common divisor of $\mathcal{L}(\Lambda)$. Then,

1. Λ lies in an infinite chain if and only if $d \neq 1$.
2. If $d = 1$, then the descendant of Λ with largest genus is the numerical semigroup generated by $\lambda_1, \dots, \lambda_{c-g-1}$.
3. If $d \neq 1$ and d is not prime, then Λ lies in infinitely many infinite chains.
4. If d is a prime then the number of infinite chains in which Λ lies is one plus the number of descendants of $\{0, \frac{\lambda_1}{d}, \dots, \frac{\lambda_{c-g-1}}{d}\} \cup \{l \in \mathbb{N}_0 : l \geq \lceil \frac{c}{d} \rceil\}$ beyond $\frac{\lambda_{c-g-1}}{d}$.

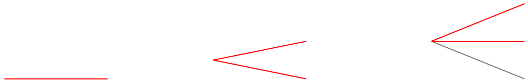
Minority of semigroups in infinite chains

Corollary. If the numerical semigroup Λ lies in an infinite chain, then it has at most two children in an infinite chain.

And we explicitly know that the only two possible children of a semigroup Λ that have infinitely many descendants are $\Lambda \setminus \{c\}$ or $\Lambda \setminus \{c+1\}$.

Proposition. Except for $\mathbb{N}_0 \setminus \{1\}$ and hyperelliptic semigroups, every numerical semigroup that is in an infinite chain has at least one child that is not.

Definition. A numerical semigroup is *fertile* if most of its children are in infinite chains.



Proposition. A fertile numerical semigroup Λ of genus $g > 2$ has three children if and only if $\Lambda = \{4k : k \geq 0\} \cup [4n+2, \infty)$ for some $n \geq 1$.

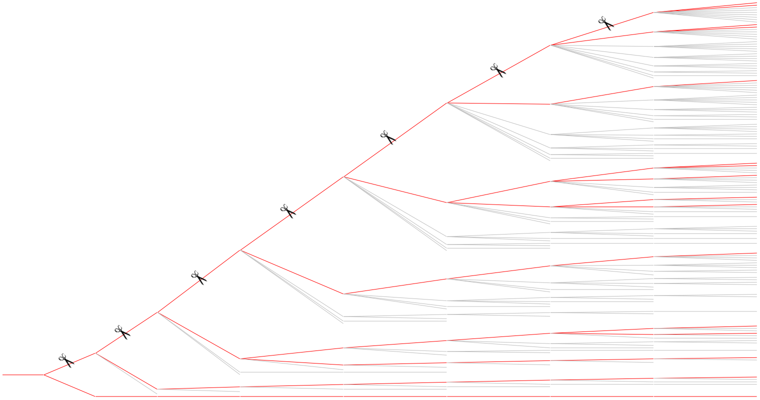
Theorem. The unique fertile semigroups of genus g are:

- $\mathbb{N}_0$ if $g = 0$.
- $\mathbb{N}_0 \setminus \{1\}$ if $g = 1$.
- H_g if $g > 1$ and $g \neq 1 \mod 3$.
- H_g and $M_{\frac{g-1}{3}}$ if $g > 1$ and $g = 1 \mod 3$.

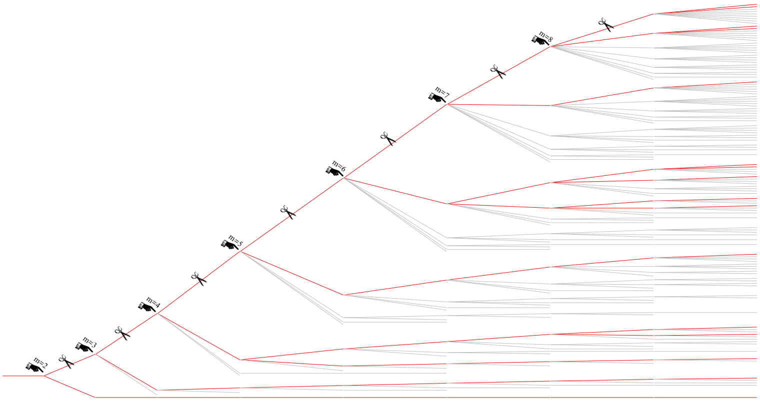
$$i_g := \#\{\Lambda : g(\Lambda) = g \text{ and } \Lambda \text{ lies in an infinite chain}\}$$

Theorem. For $g \geq 5$ we have $i_g < \dfrac{n_g}{2}$.

Fixing the multiplicity



Numerical semigroups tree up to genus 9, with infinite chains highlighted.



Numerical semigroups tree up to genus 9, with infinite chains highlighted.

Definition. The *push of* Λ, with enumeration λ, *by its multiplicity* is the numerical semigroup $\lambda_1 \oplus \Lambda := \{0\} \cup \{\lambda_1 + \lambda_j ; \lambda_j \in \Lambda\}$.

Corollary. Let Λ be a numerical semigroup with enumeration λ and let Π be a non-ordinary numerical semigroup. Then,

- If λ_k is a (effective) generator of Λ, then $\lambda_1 + \lambda_k$ is a (effective) generator of $\lambda_1 \oplus \Lambda$.
- Π is a child of Λ if and only if $\lambda_1 \oplus \Pi$ is a child of $\lambda_1 \oplus \Lambda$.
- Λ lies in an infinite chain if and only if so does $\lambda_1 \oplus \Lambda$.

$$\lambda_1 \oplus^n \Lambda := \underbrace{\lambda_1 \oplus (\cdots (\lambda_1 \oplus (\lambda_1 \oplus \Lambda)))}_{n \text{ times}}, n \geq 2$$

Prime multiplicity

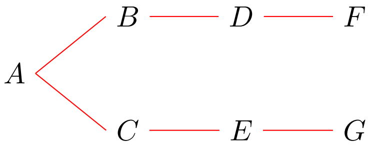
For each prime multiplicity there is only one infinite chain containing semigroups of that multiplicity. Indeed, this unique chain is $\omega(m, \mathbb{N}_0)$.

Theorem. If m is a prime, then the number $i_g(m)$ of numerical semigroups of genus g and multiplicity m that are in an infinite chain is:

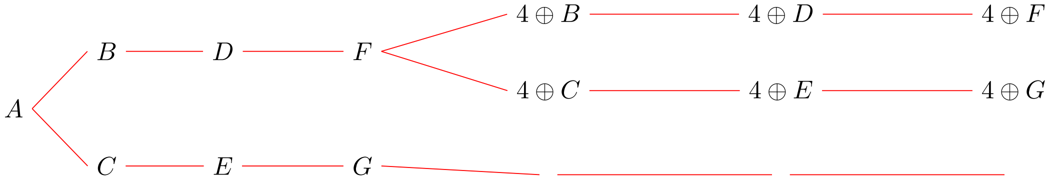
- 0, if $g < m - 1$,
- 1, otherwise.

Multiplicity 4

Tree structure of numerical semigroups of multiplicity 4 that are in an infinite chain, from genus 4 to 7.



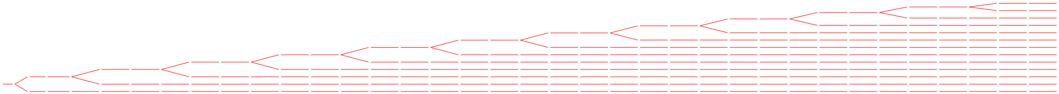
Tree structure of numerical semigroups of multiplicity 4 that are in an infinite chain, from genus 4 to 10.



Theorem. The number $i_g(m = 4)$ of numerical semigroups of genus g and multiplicity 4 that are in an infinite chain is:

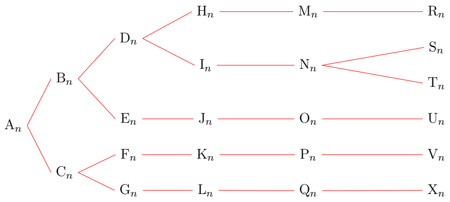
- 0, if $g \leq 2$
- 1 if $g = 3, 4$
- $\left\lfloor \dfrac{g+1}{3} \right\rfloor$, if $g \geq 5$.

Tree of numerical semigroups in infinite chains, with multiplicity 4, from genus 3 up to 40.



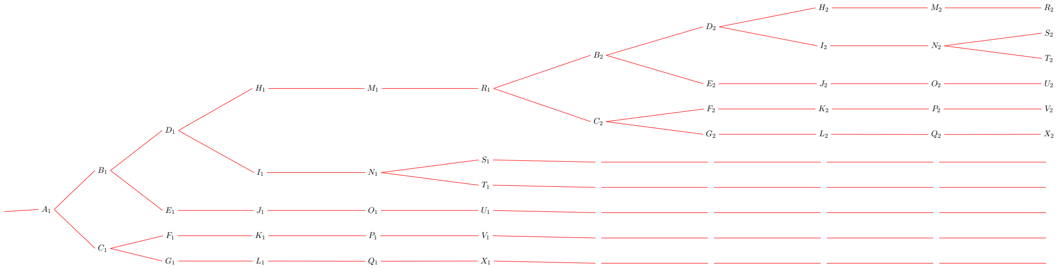
Multiplicity 6

Tree structure which is replicated on numerical semigroups tree in an infinite chain with multiplicity 6

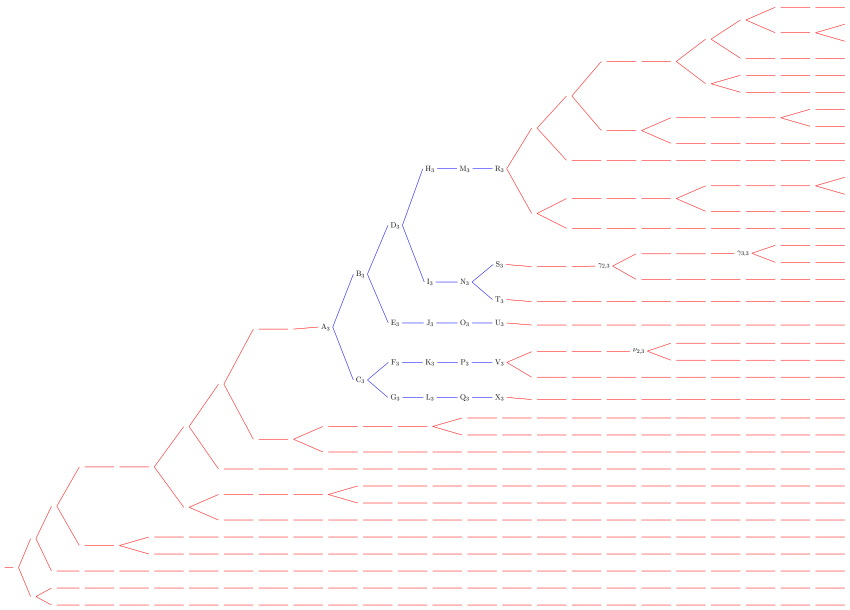


- T_n, U_n, X_n lie to 1 infinite chain
- S_n, V_n lie to n infinite chains

N. semigroups in infinite chains, with multiplicity 6, from genus 5 up to 16



N. semigroups in infinite chains, multiplicity 6, from genus 5 up to genus 31



Theorem. The number $i_g(m=6)$ of numerical semigroups of genus g and multiplicity 6 that are in an infinite chain is:

$$\begin{cases} 0 & \text{if } g \leq 4 \\ 1 & \text{if } g = 5 \\ g-4 & \text{if } 5 < g \leq 15 \text{ and } (g \mod 5) > 2 \\ g-5 & \text{if } 5 < g \leq 15 \text{ and } (g \mod 5) \leq 2 \\ g-4+\sum_{n=1}^{\lfloor\frac{g-8}{9}\rfloor} n + \sum_{n=1}^{\lfloor\frac{g-11}{9}\rfloor} n + \sum_{n=\lfloor\frac{g+1}{9}\rfloor}^{\lfloor\frac{g-12}{5}\rfloor} \lfloor\frac{g-8-5n}{4}\rfloor + \sum_{n=\lfloor\frac{g-2}{9}\rfloor}^{\lfloor\frac{g-15}{5}\rfloor} \lfloor\frac{g-11-5n}{4}\rfloor & \text{if } g \geq 16 \text{ and } (g \mod 5) > 2 \\ g-5+\sum_{n=1}^{\lfloor\frac{g-8}{9}\rfloor} n + \sum_{n=1}^{\lfloor\frac{g-11}{9}\rfloor} n + \sum_{n=\lfloor\frac{g+1}{9}\rfloor}^{\lfloor\frac{g-12}{5}\rfloor} \lfloor\frac{g-8-5n}{4}\rfloor + \sum_{n=\lfloor\frac{g-2}{9}\rfloor}^{\lfloor\frac{g-15}{5}\rfloor} \lfloor\frac{g-11-5n}{4}\rfloor & \text{if } g \geq 16 \text{ and } (g \mod 5) \leq 2 \end{cases}$$

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Muito obrigada!

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