On discrete Brunn-Minkowski type inequalities

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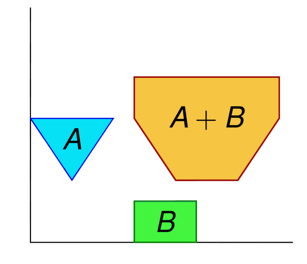
Ingredients in Brunn-Minkowski's inequality

- The Minkowski addition $A+B$ of two sets $A, B \subset \mathbb{R}^n$ is

$$A+B=\{a+b : a \in A, b \in B\}$$



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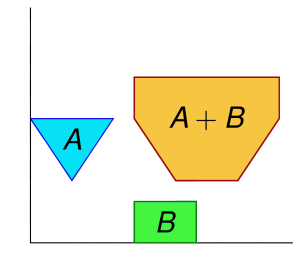

- $\mathrm{vol}(K)$ = volume (Lebesgue measure) of $K \subset \mathbb{R}^n$.

The Brunn-Minkowski inequality

Relating the volume with the Minkowski addition of compact sets (not necessarily convex), one is led to the famous Brunn-Minkowski inequality:

Brunn-Minkowski's inequality

Let $K, L \subset \mathbb{R}^n$ be non-empty compact sets. Then

$$\mathrm{vol}(K + L)^{1/n} \geq \mathrm{vol}(K)^{1/n} + \mathrm{vol}(L)^{1/n}.$$

If $\mathrm{vol}(K)\mathrm{vol}(L) > 0$ then equality holds if and only if K and L are homothetic convex bodies.

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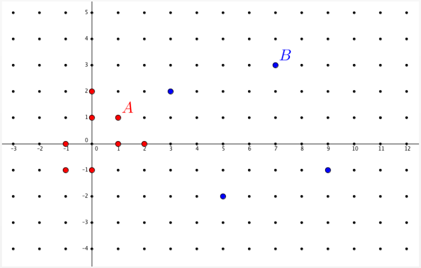

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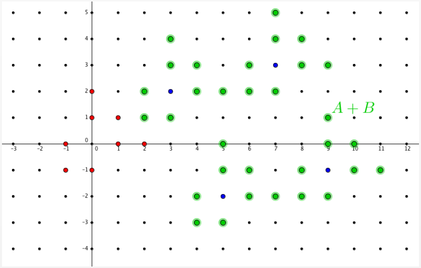

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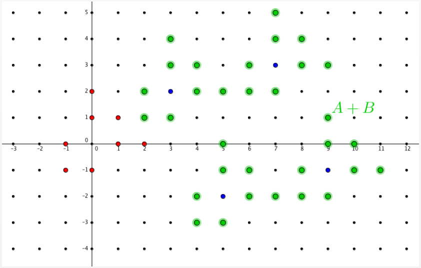

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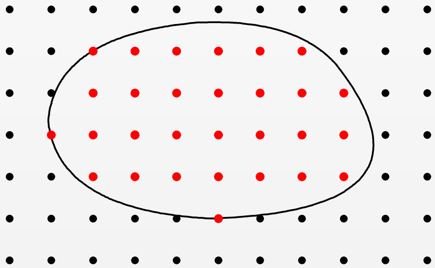
- We consider **finite subsets** $A, B \subset \mathbb{Z}^n$ **of integer points** (with the Minkowski addition), being now our measure the **cardinality** $|\cdot|$.
- Or we work with $K, L \subset \mathbb{R}^n$ **convex bodies** and our way of measuring will be the **lattice point enumerator** $\mathrm{G}_n(\cdot) = |\cdot \cap \mathbb{Z}^n|$.



Is there a classical discrete B-M inequality for $|\cdot|$?

Does a discrete Brunn-Minkowski inequality exist in the classical form for the cardinality? Namely, is it true that

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}$$

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NO! (in general)

For $A = \{0\}$ and any $B \subset \mathbb{Z}^n$ (finite),

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Therefore, a discrete Brunn-Minkowski type inequality should,

- either have a **different structure**,
- or involve **modifications of the sets**.

A discrete B-M inequality by Gardner-Gronchi

Gardner&Gronchi, 2001. A discrete analogue of $\mathrm{vol}(K+L) \geq \mathrm{vol}(B_K+B_L)$:

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$$|A+B| \geq \left|D^B_{|A|} + D^B_{|B|}\right|.$$

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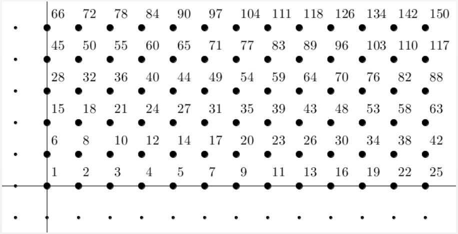
- $D^B_{|A|}$ = B-initial segment associated to A: for $m \in \mathbb{N}$, D^B_m is the set of the first m points of $\mathbb{Z}^n_{\geq 0}$ in the "B-order".



$|B| = 6$

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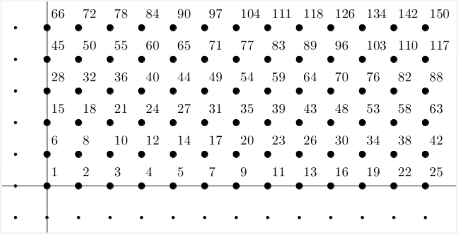
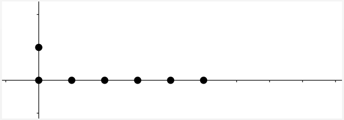


$|B| = 6$ and $m = 7$



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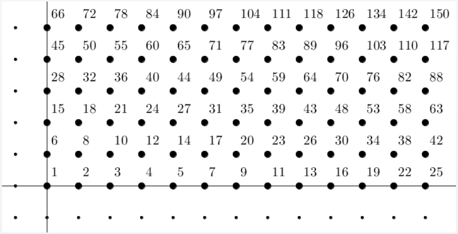

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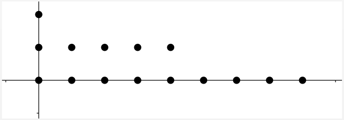


A discrete B-M inequality extending one of the sets

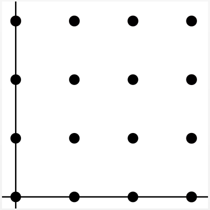
An alternative to get a "classical" B-M inequality might be to transform (one of) the sets involved in the problem, adding extra points.

H.C., Iglesias, Yepes Nicolás, 2018

Let $A, B \subset \mathbb{Z}^n$ be finite, $A, B \neq \emptyset$. Then

$$\left|\bar{A} + B\right|^{1/n} \geq |A|^{1/n} + |B|^{1/n},$$

where $\bar{A}$ is a suitably defined extension of A (not depending on B). Equality holds when A and B are lattice cubes.

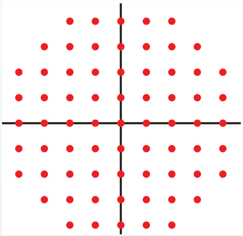


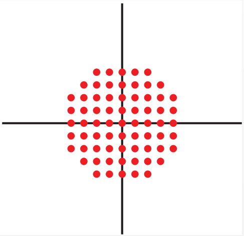
Lattice cube: $\mathcal{C}_r = [0, r]^n \cap \mathbb{Z}^n$.

What about a convex combination?

If $A, B \subset \mathbb{R}^n$ are non-empty finite sets, then

$$\left|(1-\lambda)A + \lambda B\right| \geq \left((1-\lambda)|A|^{1/n} + \lambda|B|^{1/n}\right)^n.$$





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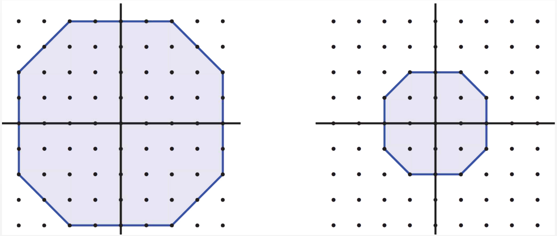

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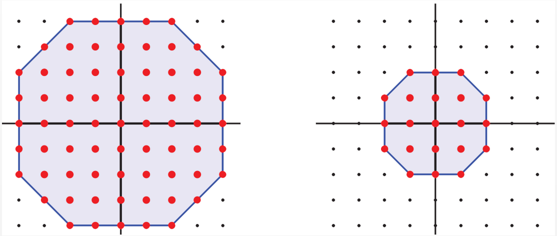

- $K, L \subset \mathbb{R}^n$ convex bodies.
- $\mathrm{G}_n(\cdot)$, the lattice point enumerator

Is there a classical discrete B-M inequality for $\mathrm{G}_n(\cdot)$?

Does a discrete Brunn-Minkowski inequality exist in the classical form for the lattice point enumerator? Namely, is it true that

$$\mathrm{G}_n\big((1-\lambda)K+\lambda L\big)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n}+\lambda\mathrm{G}_n(L)^{1/n} \text{ ?}$$

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NO! For $K=\{0\}$ and $L=[0,m]^n$, with $m\in\mathbb{N}$ odd, and $\lambda=1/2$.

$$\mathrm{G}_n\left(\frac{1}{2}K+\frac{1}{2}L\right)^{1/n}=\frac{m+1}{2}<\frac{m+2}{2}=\frac{1}{2}\mathrm{G}_n(K)^{1/n}+\frac{1}{2}\mathrm{G}_n(L)^{1/n}.$$

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Question

Given convex bodies $K,L\subset\mathbb{R}^n$, what is the "best" way to define a set M such that $(1-\lambda)K+\lambda L\subset M$ and

$$\mathrm{G}_n(M)^{1/n}\geq(1-\lambda)\mathrm{G}_n(K)^{1/n}+\lambda\mathrm{G}_n(L)^{1/n}$$

holds for all $\lambda\in(0,1)$?

Discrete B-M inequalities for $\mathrm{G}_n(\cdot)$

Iglesias, Yepes Nicolás, Zvavitch, 2020

Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets and let $\lambda \in (0,1)$. Then

$$\mathrm{G}_n\big((1-\lambda)K + \lambda L + (-1,1)^n\big)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}.$$

The inequality is sharp.

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Halikias, Klartag, Slomka, 2020

For $K, L \subset \mathbb{R}^n$ (non-empty) bounded sets one has

$$\mathrm{G}_n\left(\frac{K+L}{2} + (-1,0]^n\right)\mathrm{G}_n\left(\frac{K+L}{2} + [0,1)^n\right) \geq \mathrm{G}_n(K)\mathrm{G}_n(L).$$

Moving to the L_p-setting

The p-addition for $p \geq 1$

- The Minkowski sum of two convex bodies $K, L \subset \mathbb{R}^n$ can be defined via their support functions: $h_{K+L}(u) = h_K(u) + h_L(u)$.

The support function

$h_K(u) = \max\{\langle x, u\rangle : x \in K\}$, $u \in \mathbb{S}^{n-1}$

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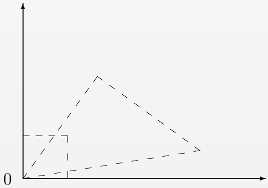
- This was extended by **Firey, 1962**: Let $p \geq 1$ and $K, L \subset \mathbb{R}^n$ be convex bodies containing the origin in their interior. The p-sum $K +_p L$ is the unique convex body such that

$$h_{K+_pL}(u) = \big(h_K(u)^p + h_L(u)^p\big)^{1/p}.$$



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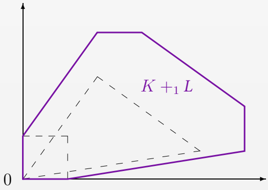

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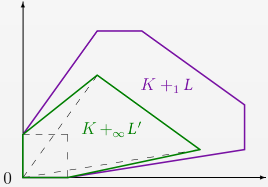

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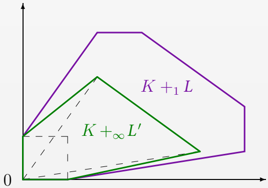
- For $1 \leq p \leq q \leq \infty$, $K +_q L \subset K +_p L$.



The p-addition. The L_p-Brunn-Minkowski inequality

Firey, 1962:

The L_p-Brunn-Minkowski inequality

Let $K, L \subset \mathbb{R}^n$ be convex bodies containing the origin in their interior, $\lambda \in (0,1)$ and $p \geq 1$. Then

$$\mathrm{vol}\big((1-\lambda)\cdot K +_p \lambda \cdot L\big)^{p/n} \geq (1-\lambda)\mathrm{vol}(K)^{p/n} + \lambda \mathrm{vol}(L)^{p/n}.$$

Here, for $r > 0$, $r \cdot K = r^{1/p} K$ is the p-scalar product. Then

$$h_{(1-\lambda)\cdot K +_p \lambda \cdot L}(u) = \Big((1-\lambda) h_K(u)^p + \lambda h_L(u)^p\Big)^{1/p}$$

The p-addition. What about $0 \leq p < 1$?

The definition of p-sum,

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is problematic when $p < 1$: it fails because the p-sum of support functions is no longer the support function of a convex body.

How can the p-addition be defined when $p < 1$?

The p-addition. What about $0 \leq p < 1$?

Any convex body K can be expressed as

$$K = \bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq h_K(u) \right\}$$

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The Wulff-shape

Let $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}^n_{\geq 0}$ be a continuous function. The Wulff-shape of f is the set

$$\mathcal{W}(f) = \bigcap_{u \in \mathbb{S}^{n-1}} \Big\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq f(u) \Big\}.$$

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Thus, for convex bodies $K, L \subset \mathbb{R}^n$ containing the origin, all $\lambda \in (0,1)$ and any $p \geq 1$,

$$(1-\lambda) \cdot K +_p \lambda \cdot L = \mathcal{W}\Big(\big((1-\lambda) h_K^p + \lambda h_L^p \big)^{1/p} \Big).$$

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Definition

For $K, L \subset \mathbb{R}^n$ convex bodies containing 0, $\lambda \in (0,1)$ and $0 \leq p < 1$,

$$(1-\lambda) \cdot K +_p \lambda \cdot L = \mathcal{W}\Big(\big((1-\lambda) h_K^p + \lambda h_L^p \big)^{1/p} \Big),$$

The p-addition. What about $0 \leq p < 1$?

The Wulff-shape

Let $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}^n_{\geq 0}$ be a continuous function. The Wulff-shape of f is the set

$$\mathcal{W}(f) = \bigcap_{u \in \mathbb{S}^{n-1}} \Big\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq f(u) \Big\}.$$

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with

$$(1-\lambda) \cdot K +_0 \lambda \cdot L = \mathcal{W}\Big(h_K^{1-\lambda} h_L^{\lambda} \Big).$$

The log-Brunn-Minkowski inequality

Böröczky, Lutwak, Yang, Zhang, 2012:

Conjecture: The log-Brunn-Minkowski inequality

Let $K, L \subset \mathbb{R}^n$ be 0-symmetric convex bodies, and let $\lambda \in (0,1)$. Then

$$\mathrm{vol}\big((1-\lambda)\cdot K +_0 \lambda \cdot L\big) \geq \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda}.$$

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More generally, one can pose the following

General conjecture

Let $K, L \subset \mathbb{R}^n$ be 0-symmetric convex bodies, $0 \leq p < 1$ and $\lambda \in (0,1)$. Then

$$\mathrm{vol}\big((1-\lambda)\cdot K +_p \lambda\cdot L\big) \geq \Big((1-\lambda)\mathrm{vol}(K)^{p/n} + \lambda\mathrm{vol}(L)^{p/n}\Big)^{n/p}.$$

A discrete L_p-Brunn-Minkowski inequality ($p \geq 1$)

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H. C., Lucas, Yepes Nicolás, 2021

Let $K, L \subset \mathbb{R}^n$ be bounded sets and let $p \geq 1$. Then, for all $\lambda \in (0,1)$,

$$\mathrm{G}_n\Big((1-\lambda)\cdot K +_p \lambda \cdot L + (-1,1)^n\Big)^{p/n} \geq (1-\lambda)\mathrm{G}_n(K)^{p/n} + \lambda \mathrm{G}_n(L)^{p/n}.$$

The inequality is sharp: the cubes $[0,m]^n$ gives equality.

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- The case of $p = 1$ corresponds to the inequality of Iglesias, Yepes Nicolás & Zvavitch, 2020.
- The discrete L_p Brunn-Minkowski type inequality implies the classical L_p-Brunn-Minkowski inequality for n-dimensional compact sets

$$\mathrm{vol}\big((1-\lambda)\cdot K +_p \lambda \cdot L\big) \geq \Big((1-\lambda)\mathrm{vol}(K)^{p/n} + \lambda \mathrm{vol}(L)^{p/n}\Big)^{n/p}.$$

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A couple of further comments:

- The cube $(-1,1)^n$ cannot be, in general, reduced by means of a smaller cube $(-1,a]^n$, $a<1$ (just take $K=[0,1], L=[0,2]\subset\mathbb{R}$).

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  would be a better inequality.

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  would be a better inequality.
  However, the Minkowski sum of the cube $(-1,1)^n$ cannot be replaced by its p-sum (just take $K=[0,1], L=[0,2]\subset\mathbb{R}$, $p=2$ and $\lambda=\frac{1}{2}$.)

A discrete log-Brunn-Minkowski inequality

H. C., Lucas, 2021

Let $K, L \subset \mathbb{R}^n$ be 0-symmetric convex bodies and let $\lambda \in (0,1)$. We write $C_n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$. If either K, L are unconditional convex bodies or $n = 2$, then

$$\mathrm{G}_n\left((1-\lambda)\cdot(K+C_n) +_0 \lambda\cdot(L+C_n)+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right) \geq \mathrm{G}_n(K)^{1-\lambda}\mathrm{G}_n(L)^{\lambda}.$$

The inequality is sharp.

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The inequality is sharp.

It implies the log-Brunn-Minkowski inequality

$$\mathrm{vol}\big((1-\lambda)\cdot K +_0 \lambda\cdot L\big) \geq \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda}$$

both for unconditional convex bodies and for $n = 2$.

A discrete log-Brunn-Minkowski inequality

$$\boxed{\mathrm{G}_n\left((1-\lambda)\cdot(K+C_n)+_0\lambda\cdot(L+C_n)+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right)\geq \mathrm{G}_n(K)^{1-\lambda}\mathrm{G}_n(L)^{\lambda}.}$$

A few further comments:

A discrete log-Brunn-Minkowski inequality

$$\mathrm{G}_n\left((1-\lambda)\cdot(K+C_n)+_0\lambda\cdot(L+C_n)+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right)\geq \mathrm{G}_n(K)^{1-\lambda}\mathrm{G}_n(L)^{\lambda}.$$

A few further comments:

- The cube C_n Minkowski-added to K and L cannot be, in general, avoided; not even summing up a bigger cube $(-\beta,\beta)^n$, $\beta > \frac{1}{2}$.

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$$\mathrm{G}_n\left((1-\lambda)\cdot(K+C_n)+_0\lambda\cdot(L+C_n)+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right)\geq \mathrm{G}_n(K)^{1-\lambda}\mathrm{G}_n(L)^{\lambda}.$$

A few further comments:

- The cube C_n Minkowski-added to K and L cannot be, in general, avoided; not even summing up a bigger cube $(-\beta,\beta)^n$, $\beta>\frac{1}{2}$.
- Similarly, the Minkowski addition of $(-\frac{1}{2},\frac{1}{2})^n$ is necessary.

A discrete L_p-Brunn-Minkowski inequality ($p < 1$)

H. C., Lucas, 2021

Let $K, L \subset \mathbb{R}^n$ be unconditional convex bodies and let $\lambda \in (0,1)$. We write $C_n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$. Then, for any $0 < p < 1$,

$$\mathrm{G}_n\left((1-\lambda)\cdot(K + C_n) +_p \lambda\cdot(L + C_n) + \left(-\frac{1}{2}, \frac{1}{2}\right)^n\right)^{p/n} \geq (1-\lambda)\mathrm{G}_n(K)^{p/n} + \lambda \mathrm{G}_n(L)^{p/n}.$$

It implies the corresponding L_p-Brunn-Minkowski inequality for unconditional convex bodies.

The End

Thank you for your attention!!