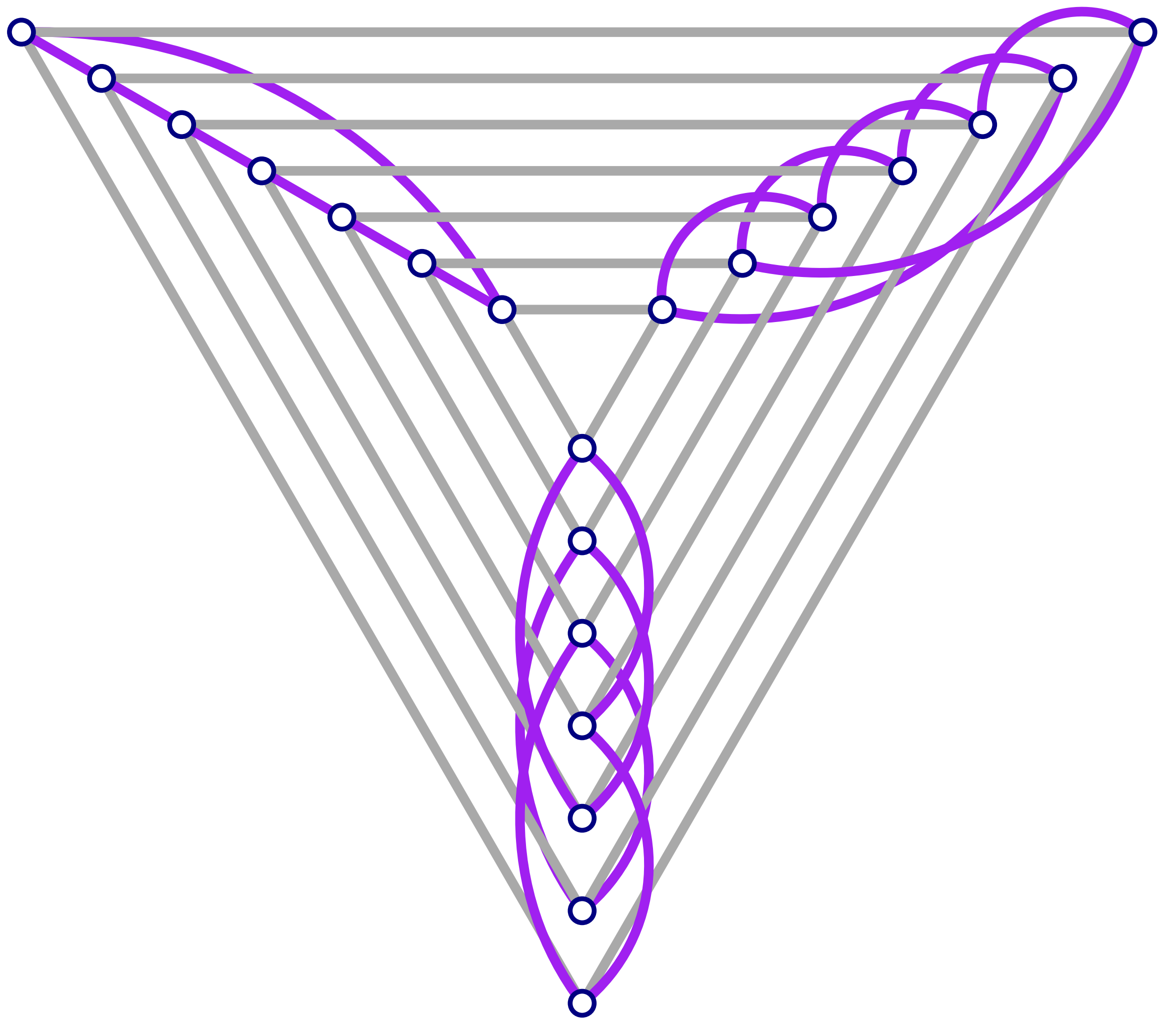
Coloring Cayley graphs

Kolja Knauer
UB, Barcelona

Ignacio García-Marco
Universidad de la Laguna



Sesiones Especiales RSME

22.02.24

CUB: Combinatorics at Universitat de Barcelona

Kolja Knauer

Arnau Padrol

Vincent Pilaud

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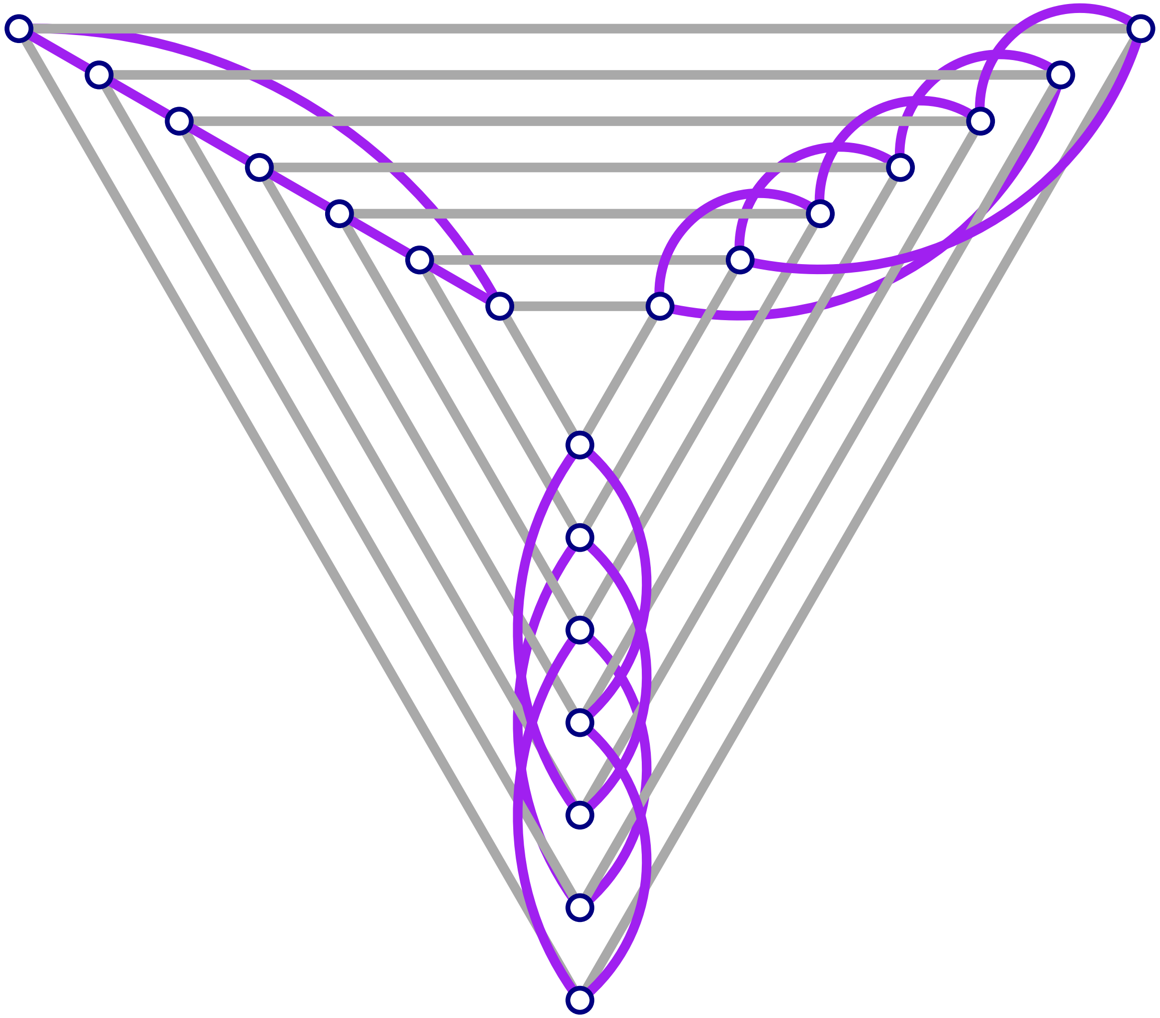

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right **Cayley graphs** of groups

G group and $C \subseteq G \rightsquigarrow$ Cay(G, C) (directed) Cayley graph

vertices $V = G$
arcs A: (a, b) for each $c \in C$ such that $a \cdot c = b$

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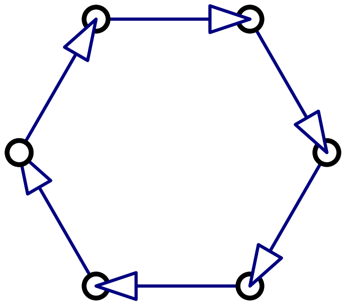
examples:

Cay$(\mathbb{Z}_6, \{1\})$



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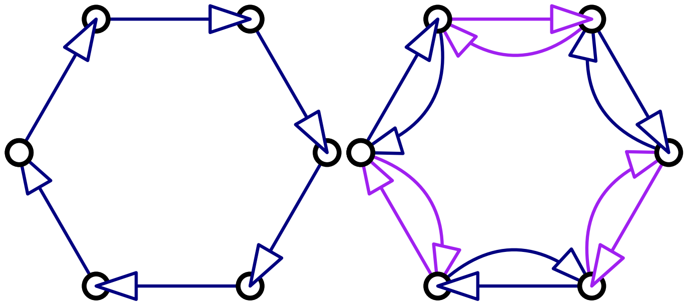

D_n=symmetry group of regular n-gon

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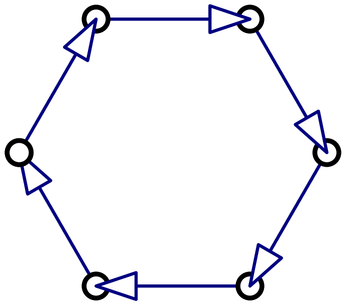

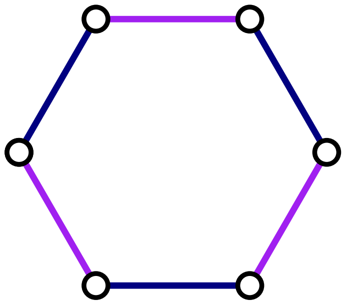

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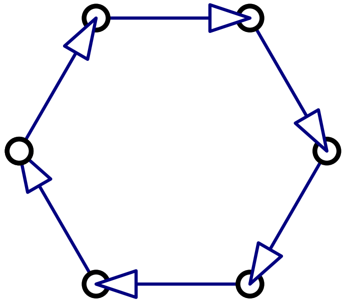

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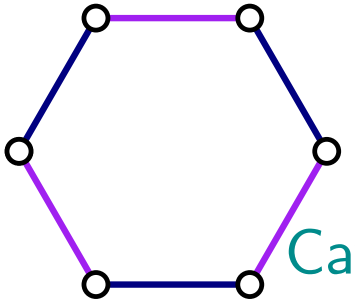

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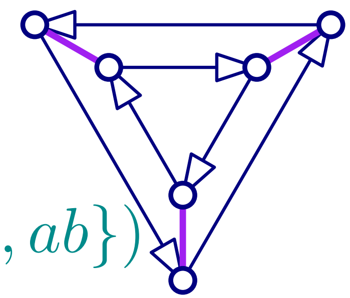

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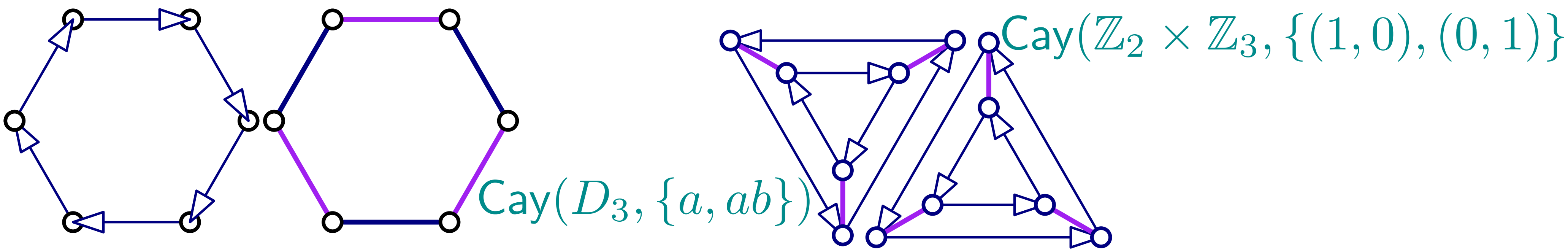

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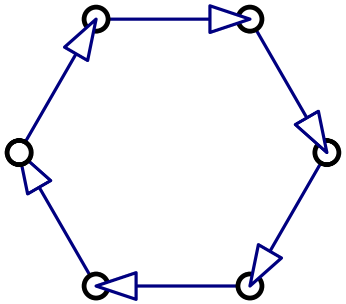

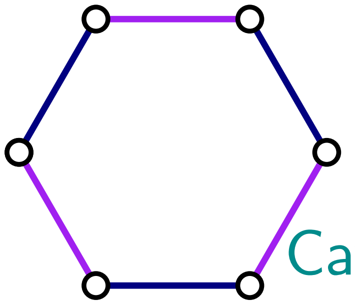

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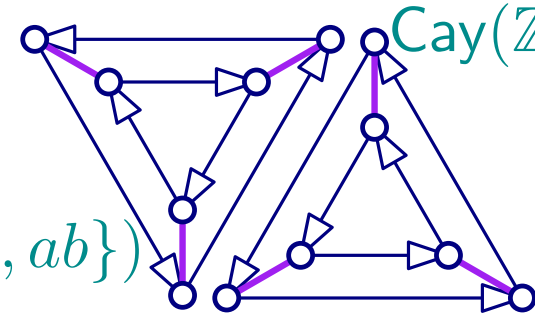
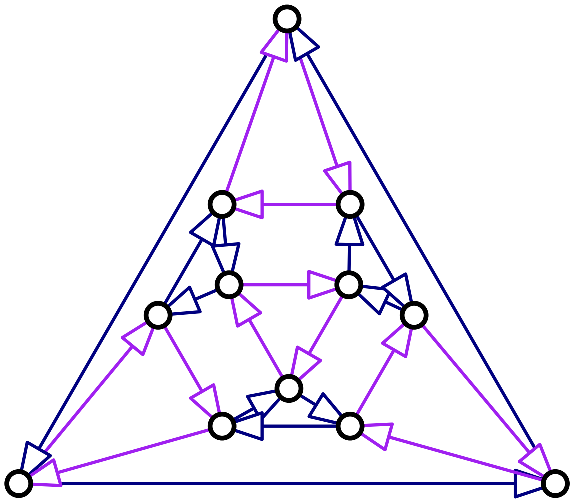


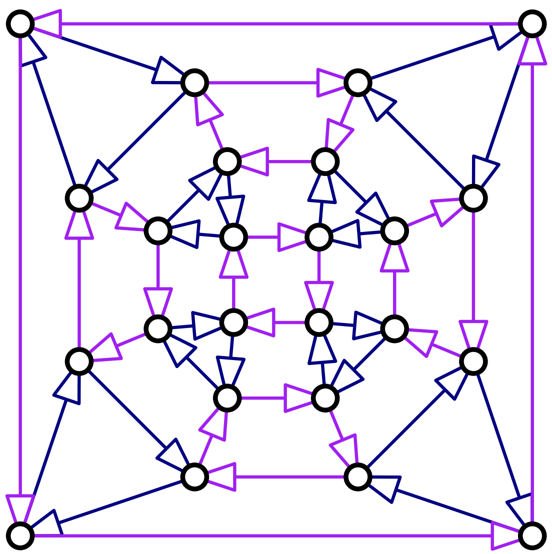
Cay$(\mathbb{Z}_2 \times \mathbb{Z}_3, \{(1, 0), (0, 1)\})$

Obs: $\mathrm{Cay}(G, C)$ connected $\iff < C >= G$ (from now on always)

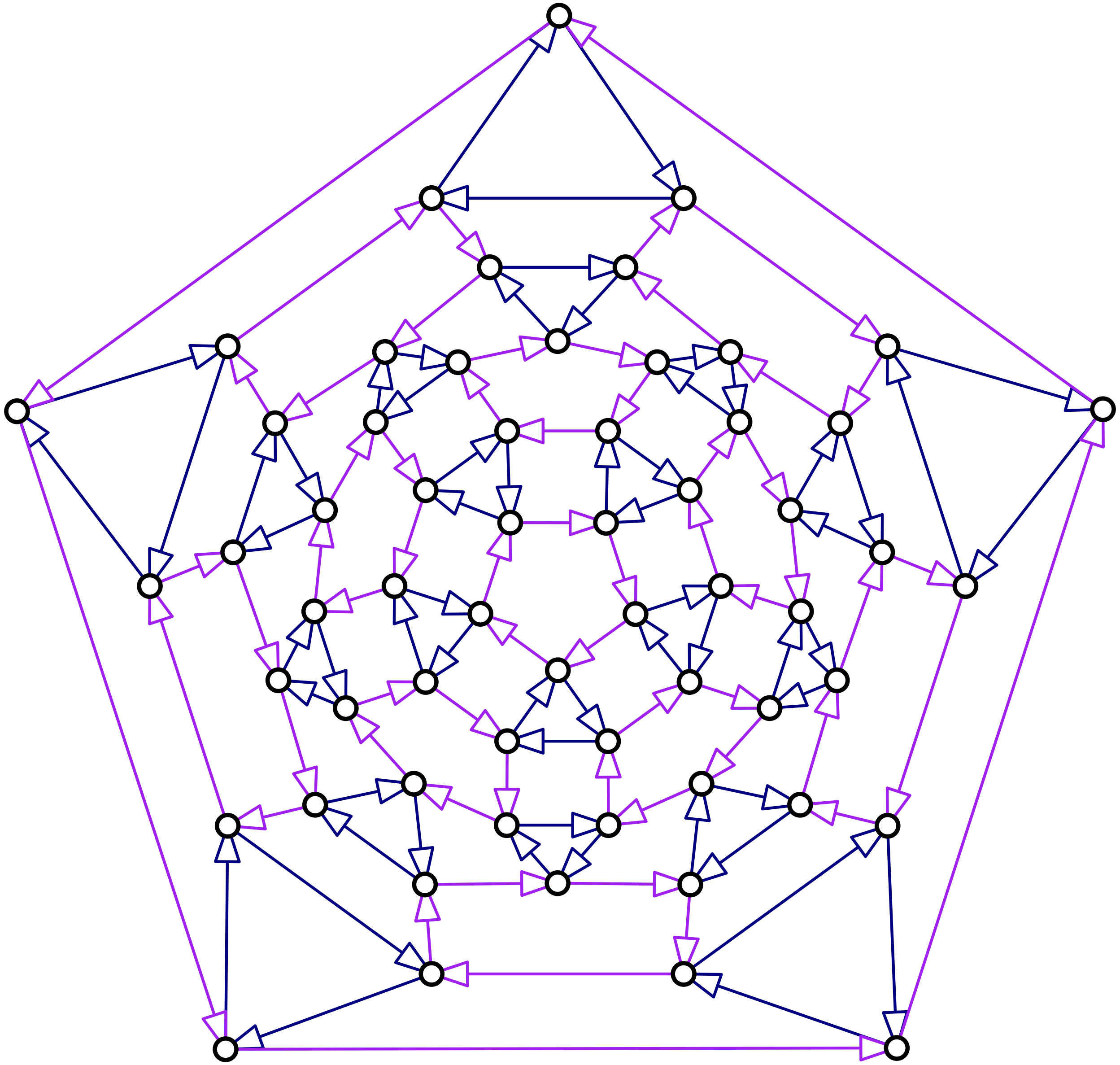
Beau-



Cay($A_4, \{(123), (234)\}$)

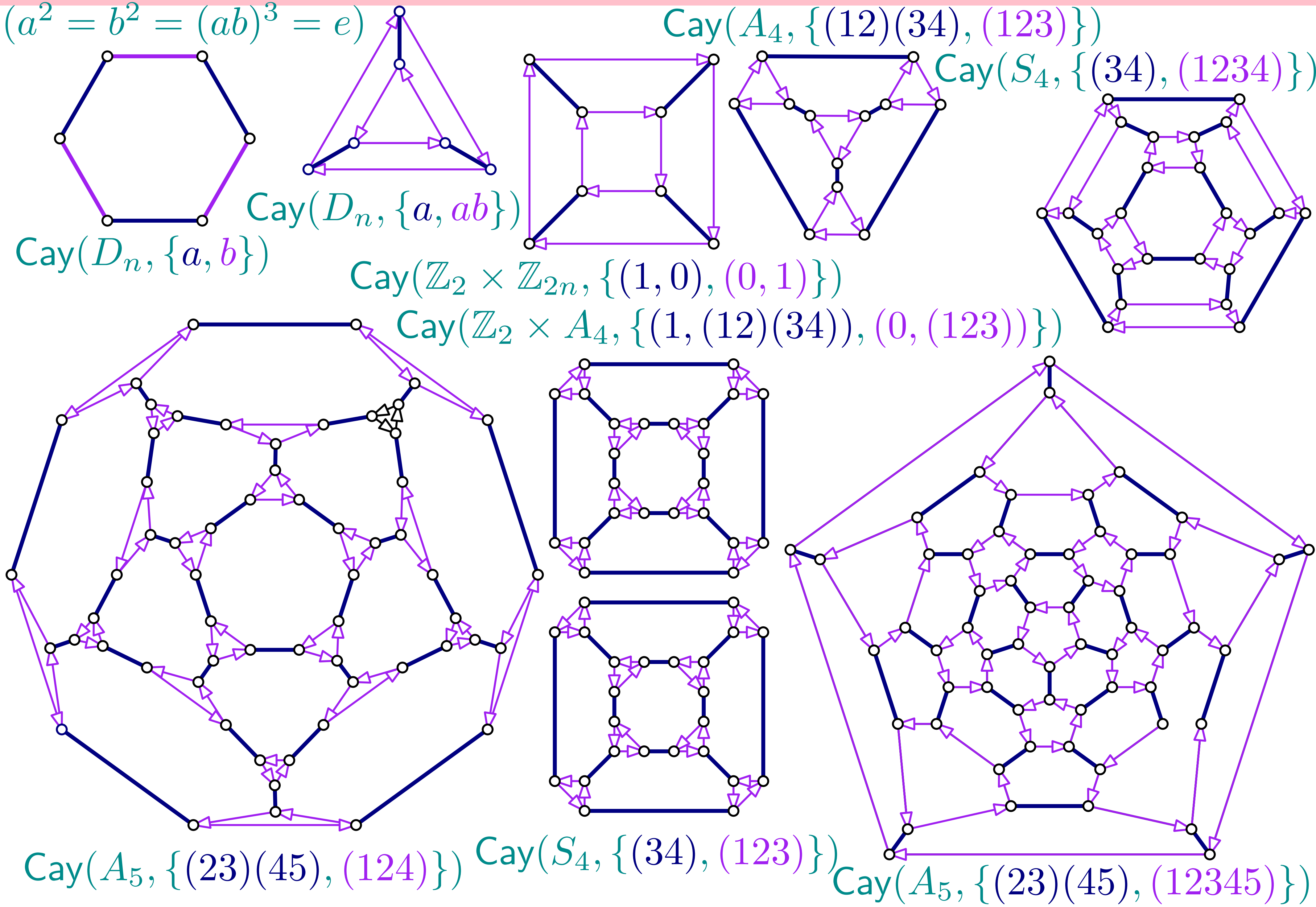


Cay($S_4, \{(123), (1234)\}$)



Cay($A_5, \{(124), (12345)\}$)

-ti-


$(a^2 = b^2 = (ab)^3 = e)$
Cay$(A_4, \{(12)(34), (123)\})$
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Cay$(D_n, \{a, ab\})$
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Cay$(\mathbb{Z}_2 \times \mathbb{Z}_{2n}, \{(1, 0), (0, 1)\})$
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Cay$(A_5, \{(23)(45), (124)\})$
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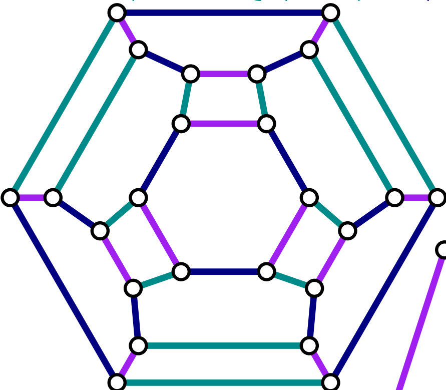
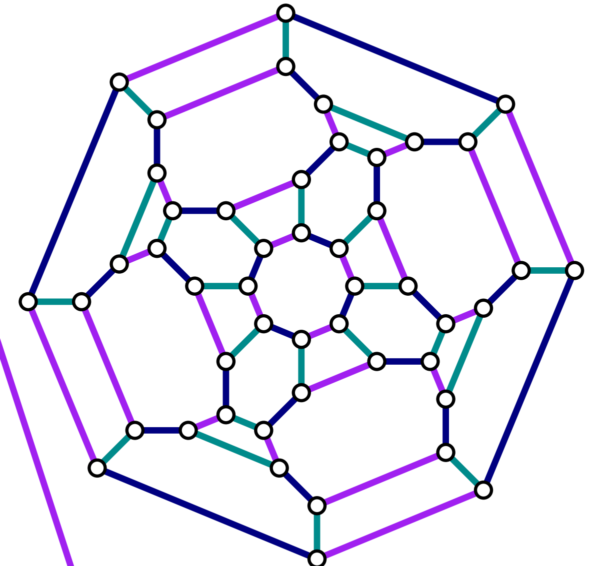
Cay$(A_5, \{(23)(45), (12345)\})$

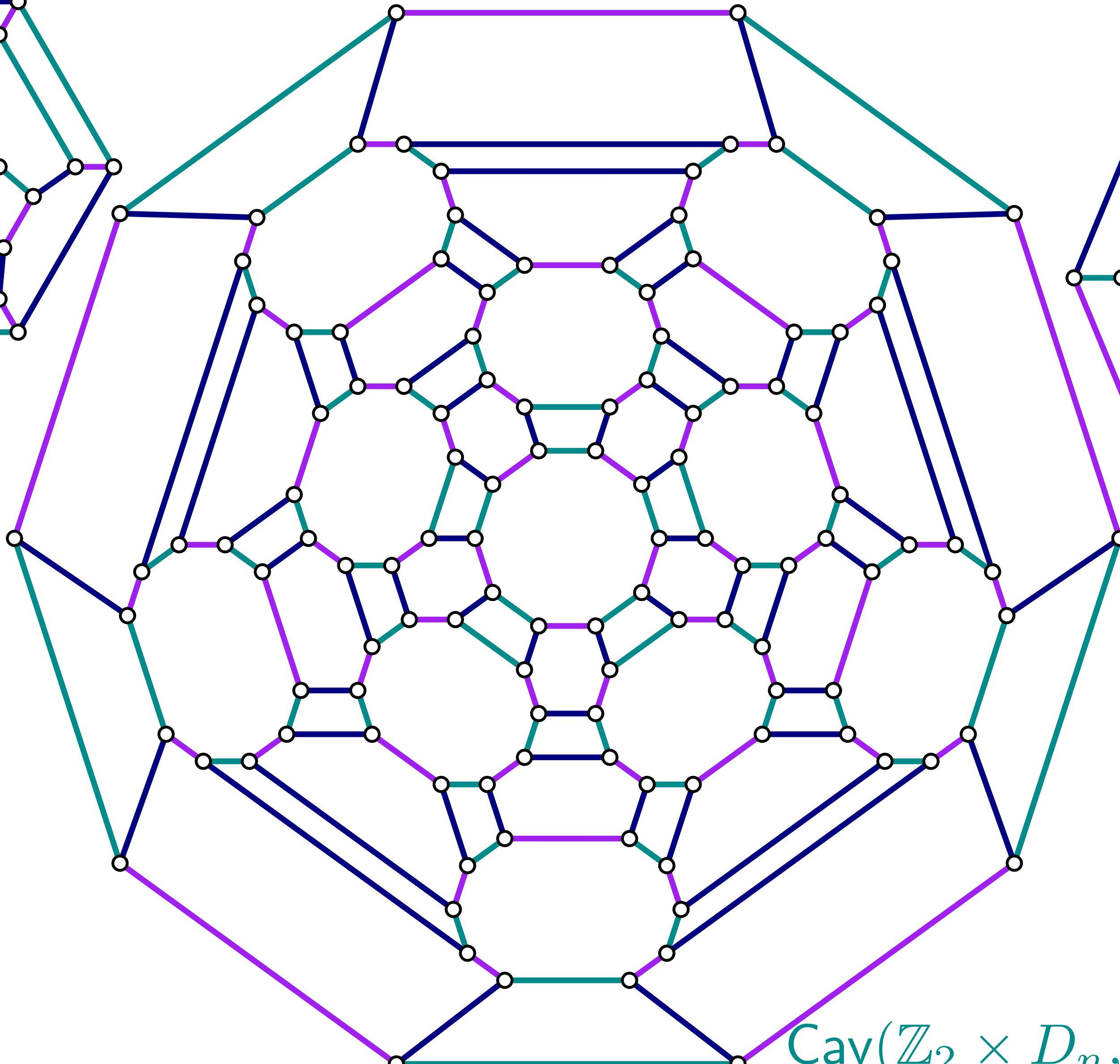
-full!

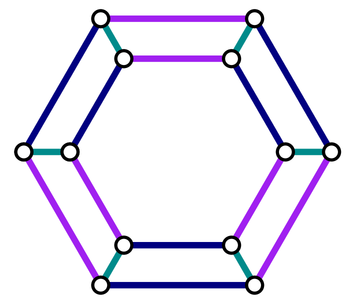
Cay$(S_4, \{(12), (23), (34)\})$



Cay$(\mathbb{Z}_2 \times S_4, \{(0, (12)), (0, (23)), (1, (12)(34))\})$







Cay$(\mathbb{Z}_2 \times D_n, \{(1, e), (0, a), (0, b)\})$

Cay$(\mathbb{Z}_2 \times A_5, \{(1, (12)(35)), (1, (24)(35)), (1, (23)(45))\})$

Coloring Cayley graphs

(ignore orientations)

Obs: clique-number ω of Cayley graphs unbounded ($\mathrm{Cay}(G,G) = K_n$)

Thm[Lubotzky, Phillips, Arnak '88]: Cayley graphs not χ-bounded (high girth expanders)

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$\mathrm{Cay}(G, C)$ is:

minimal if $< C >= G$ inclusion-minimal

semiminimal if $< c_1, \dots, c_k >= G$ and and $c_i \notin < c_1, \dots, c_{i-1} > \forall_i$

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Prop[Babai ´78]: every $\Gamma \subseteq \mathrm{Cay}(G,C)$ has edge-coloring s.th:

every vertex sees every color at most twice **and**

every cycle sees no color exactly once, if **minimal**

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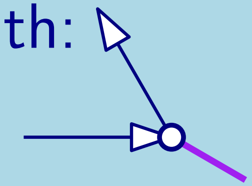

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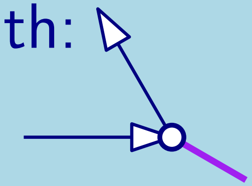

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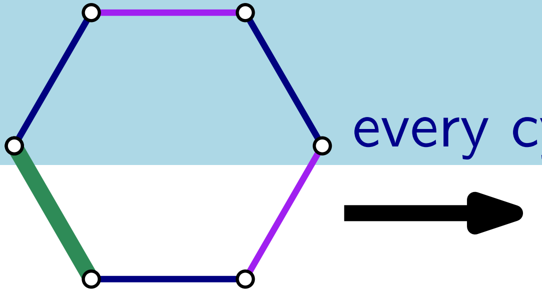

$a = c\,b\,c\,b\,c$

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Conj[Babai ´78]: $\exists M$: every *some popular color graph* has $\chi(\Gamma) \leq M$

Popular color graphs of unbounded χ
(following Tutte)

let Γ_k popular color graph with $\chi = k$ and order n

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$\rightsquigarrow \Gamma_{k+1}$:

X set of nk independent vertices

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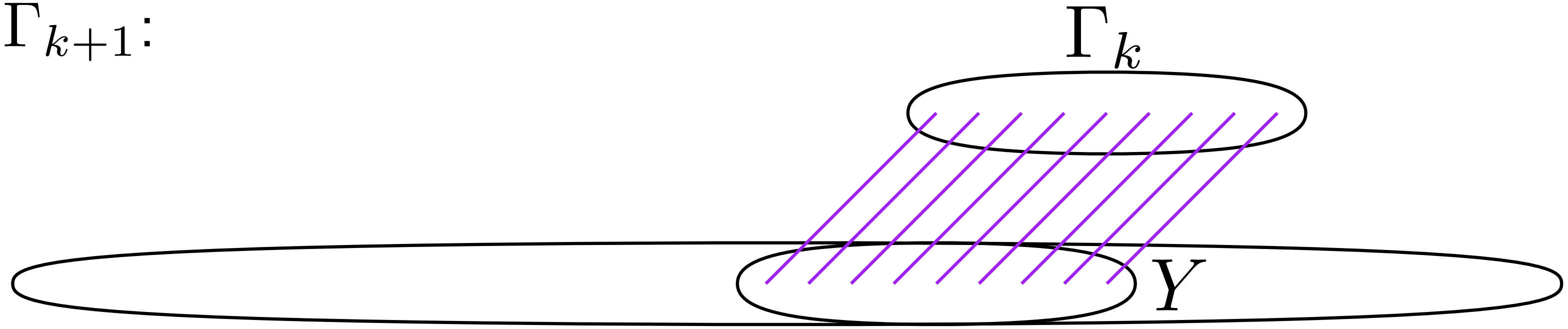

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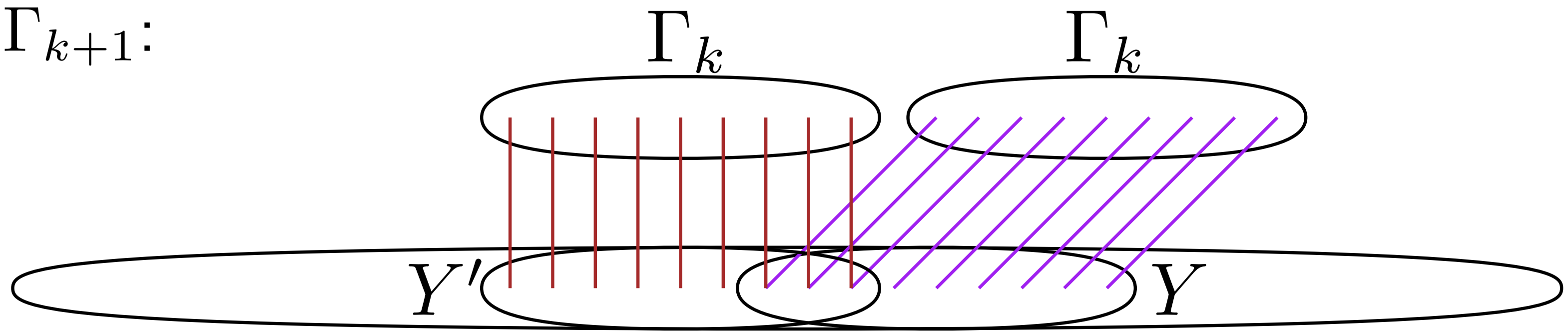

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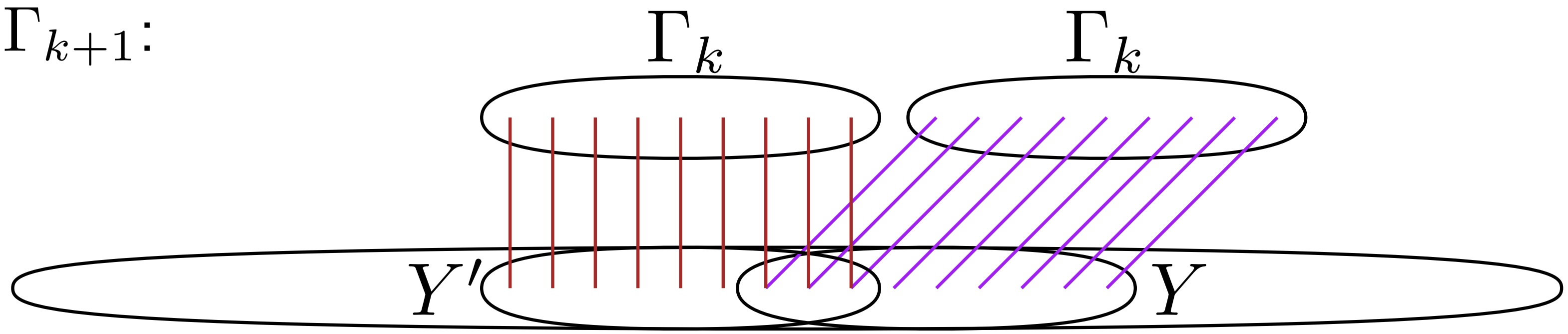

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every new cycle has to use some matching twice $\rightsquigarrow$ **popular color**

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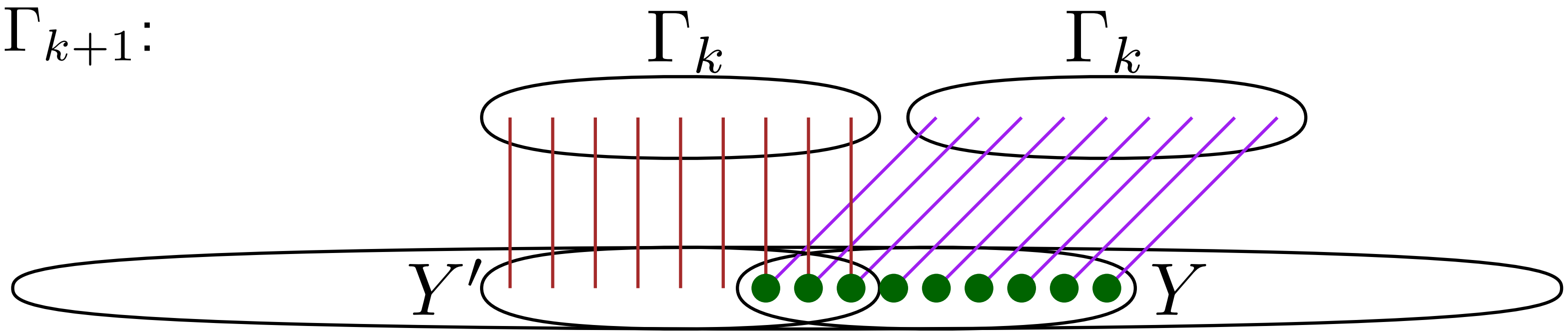

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k-coloring $f(\Gamma_{k+1}) \rightsquigarrow$ monochromatic set $Y \subseteq X$ of size $\frac{X}{k} = n$

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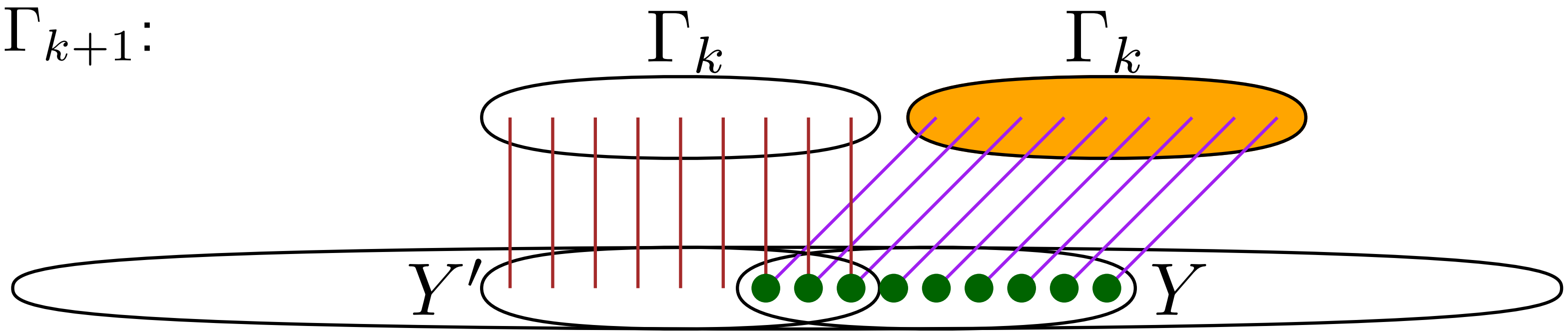

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$\rightsquigarrow$ only $k-1$ colors for Γ_k...contradiction

Popular color graphs of unbounded χ (following Tutte)

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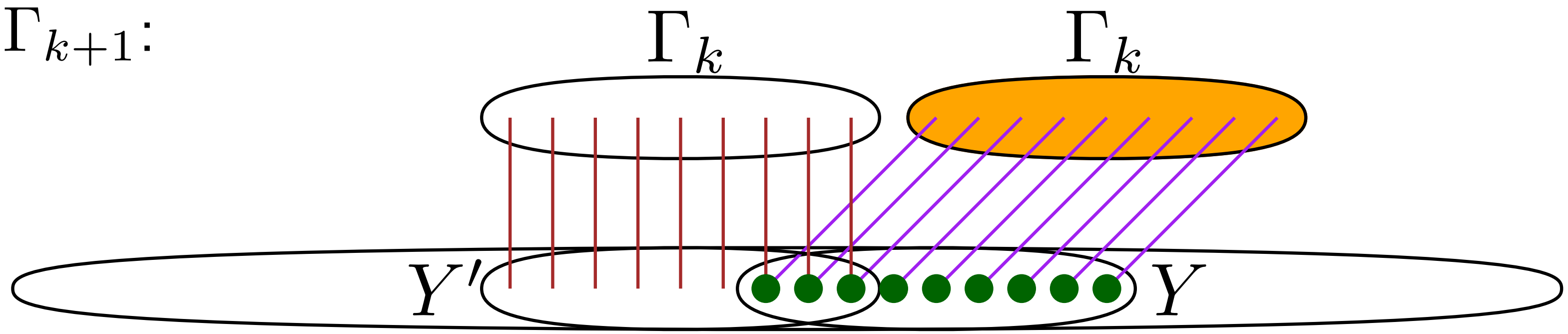
$\rightsquigarrow \Gamma_{k+1}$:



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k-coloring $f(\Gamma_{k+1}) \rightsquigarrow$ monochromatic set $Y \subseteq X$ of size $\frac{X}{k} = n$

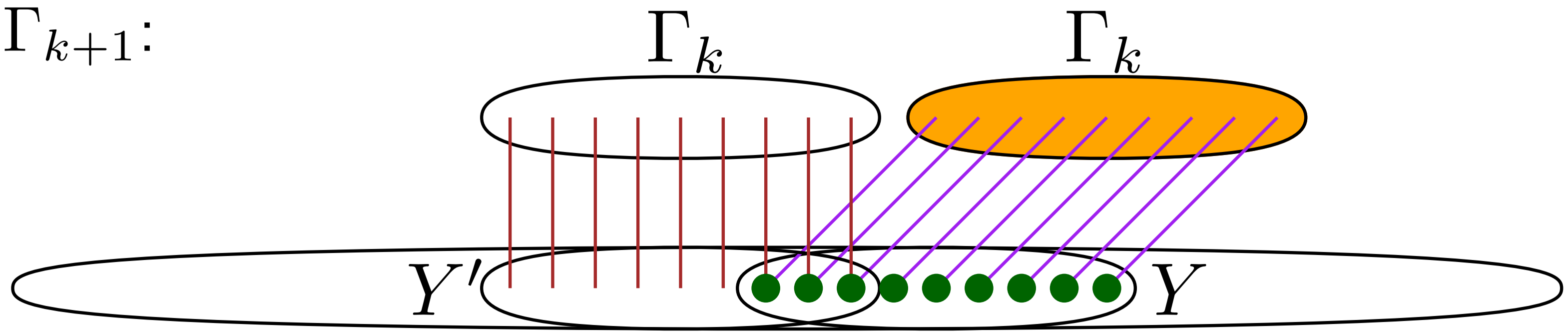
$\rightsquigarrow$ only $k-1$ colors for Γ_k...contradiction

$\rightsquigarrow \chi(\mathbf{\Gamma_{k+1}}) > \mathbf{k}$

Popular color graphs of unbounded χ
(following Tutte)

let Γ_k popular color graph with $\chi = k$ and order n

$\rightsquigarrow \Gamma_{k+1}$:



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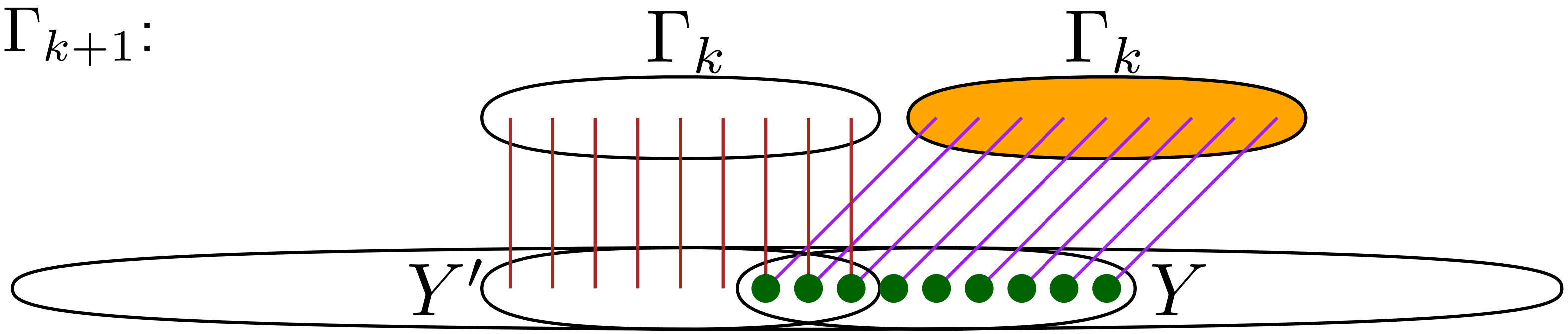

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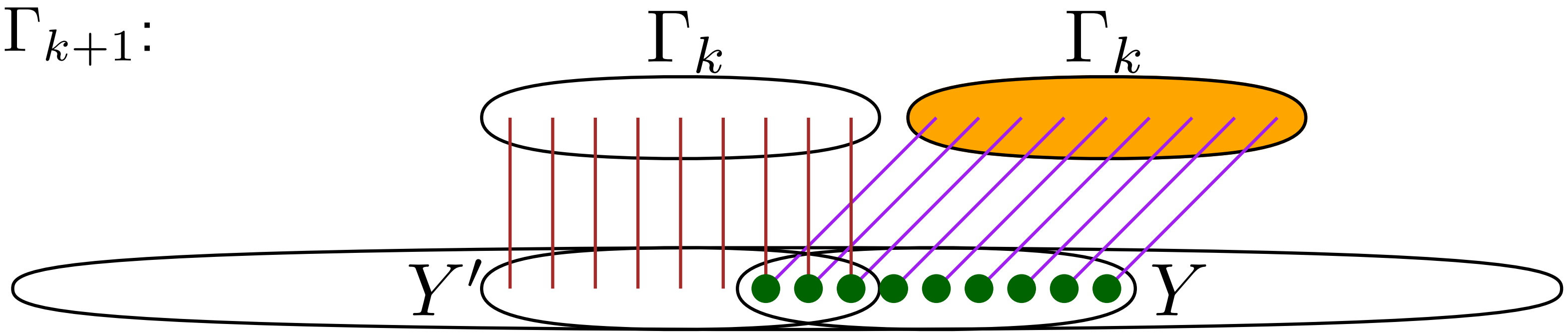

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Coloring minimal Cayley graphs

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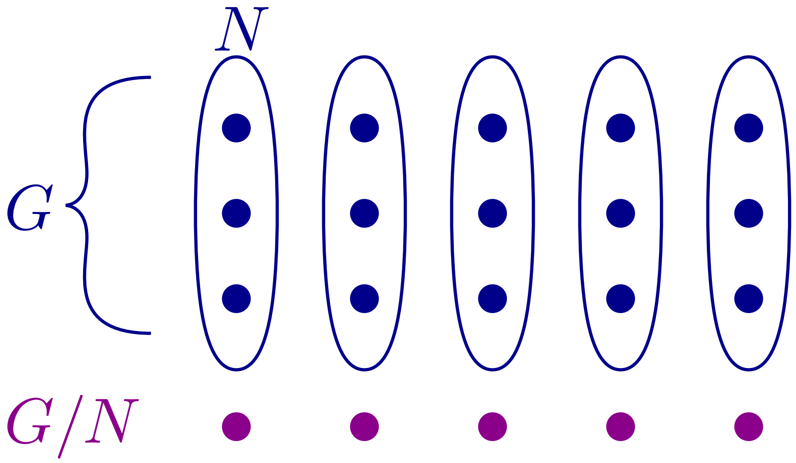

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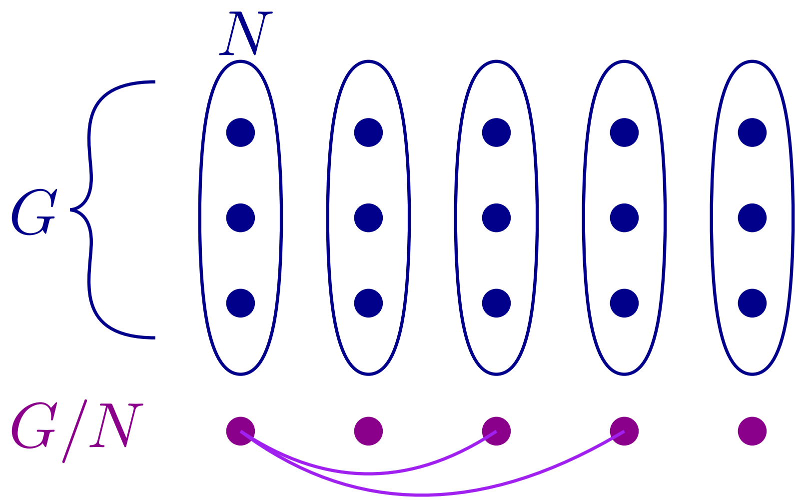

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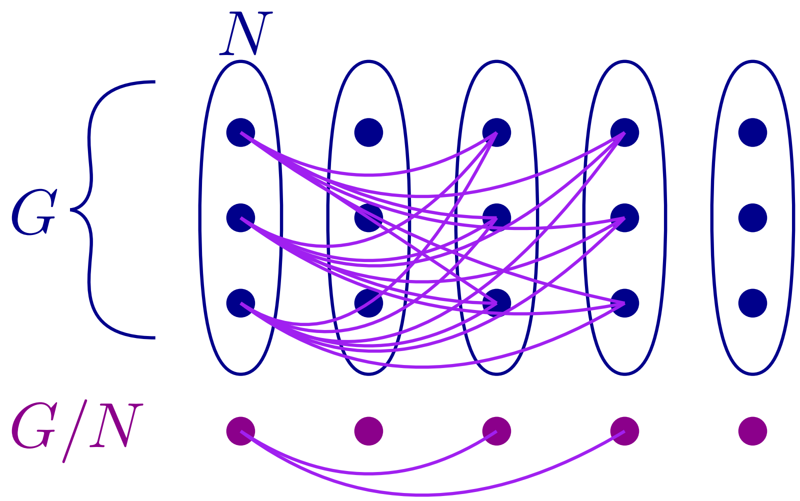

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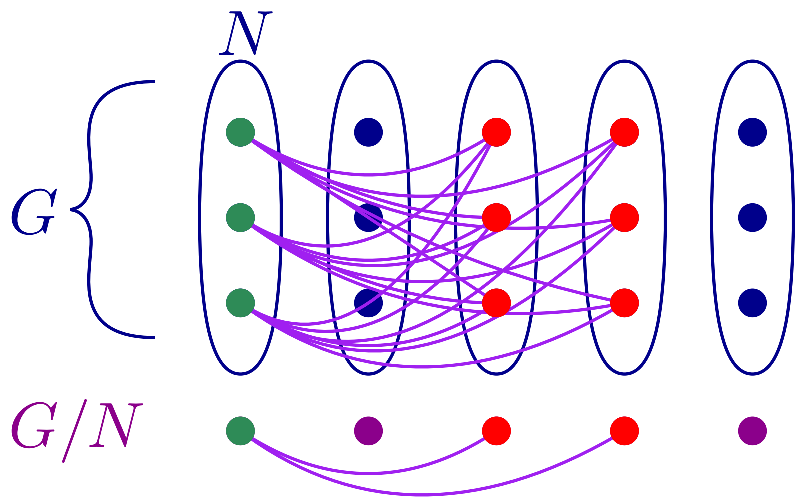

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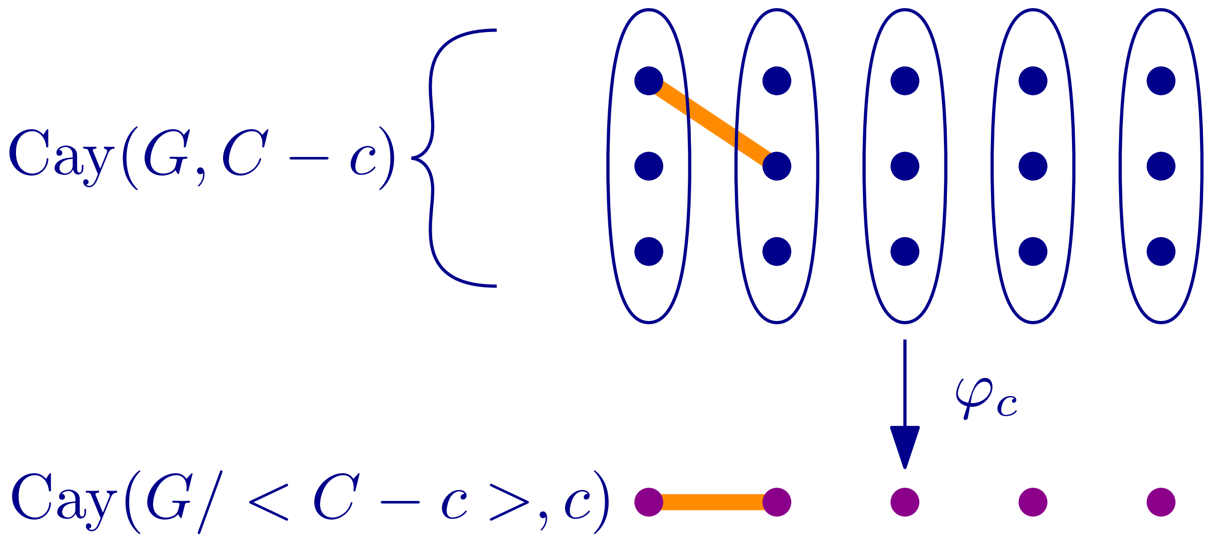

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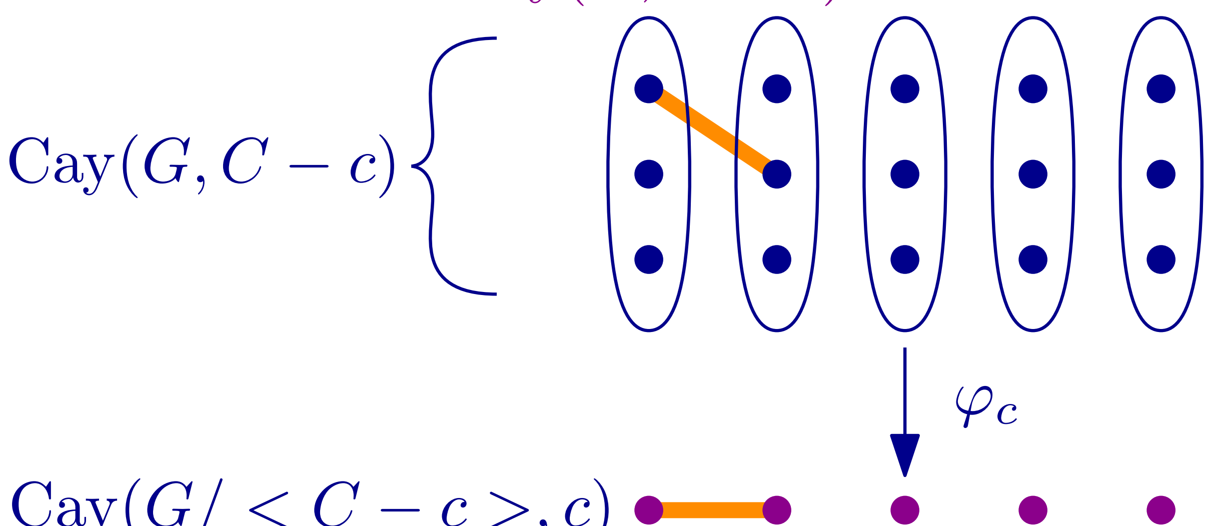

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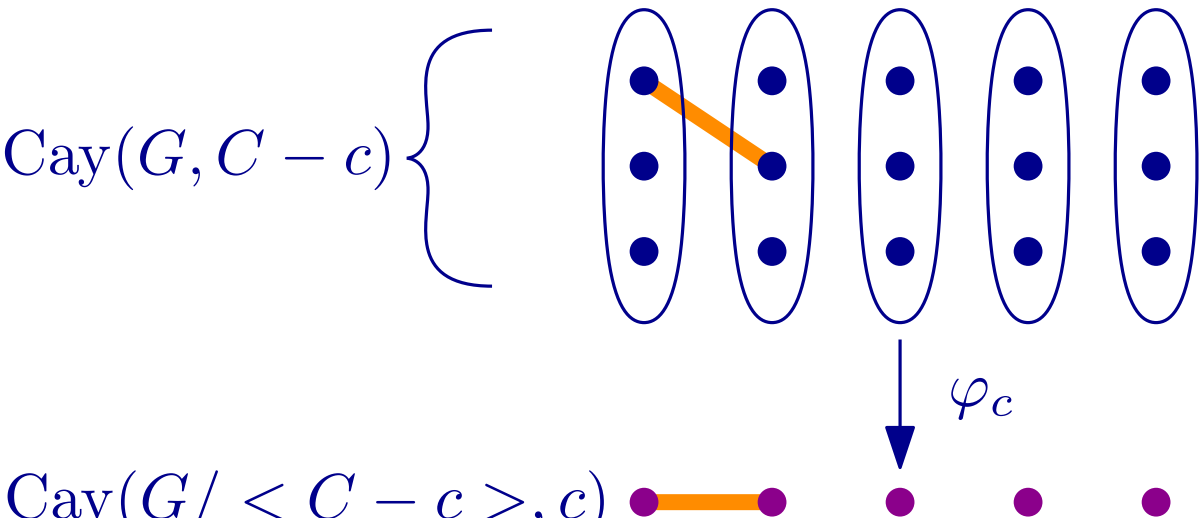

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every subgroup normal, (almost only abelian groups)

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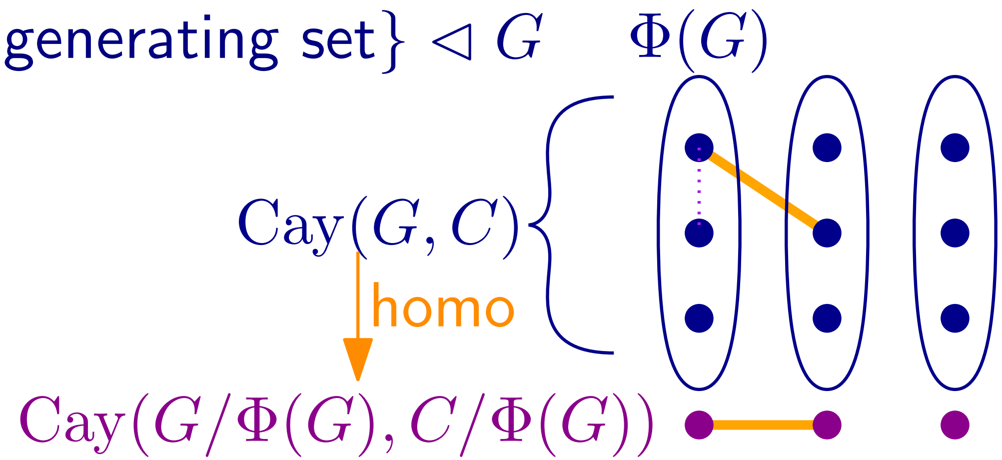

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if $c_1\Phi(G) \cdots c_k\Phi(G) = c\Phi(G)$ then
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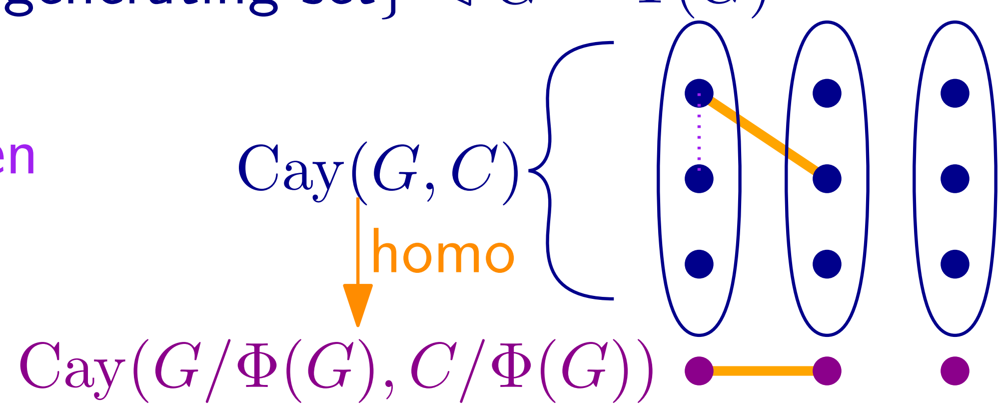
$\implies < (C - c) \cup \Phi(G) >= G$
$\rightsquigarrow$ contradiction



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↑ $G/\Phi(G)$ abelian, e.g. p-groups

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abelian $\rtimes \mathbb{Z}_2$

3–Coloring minimal Cayley graphs

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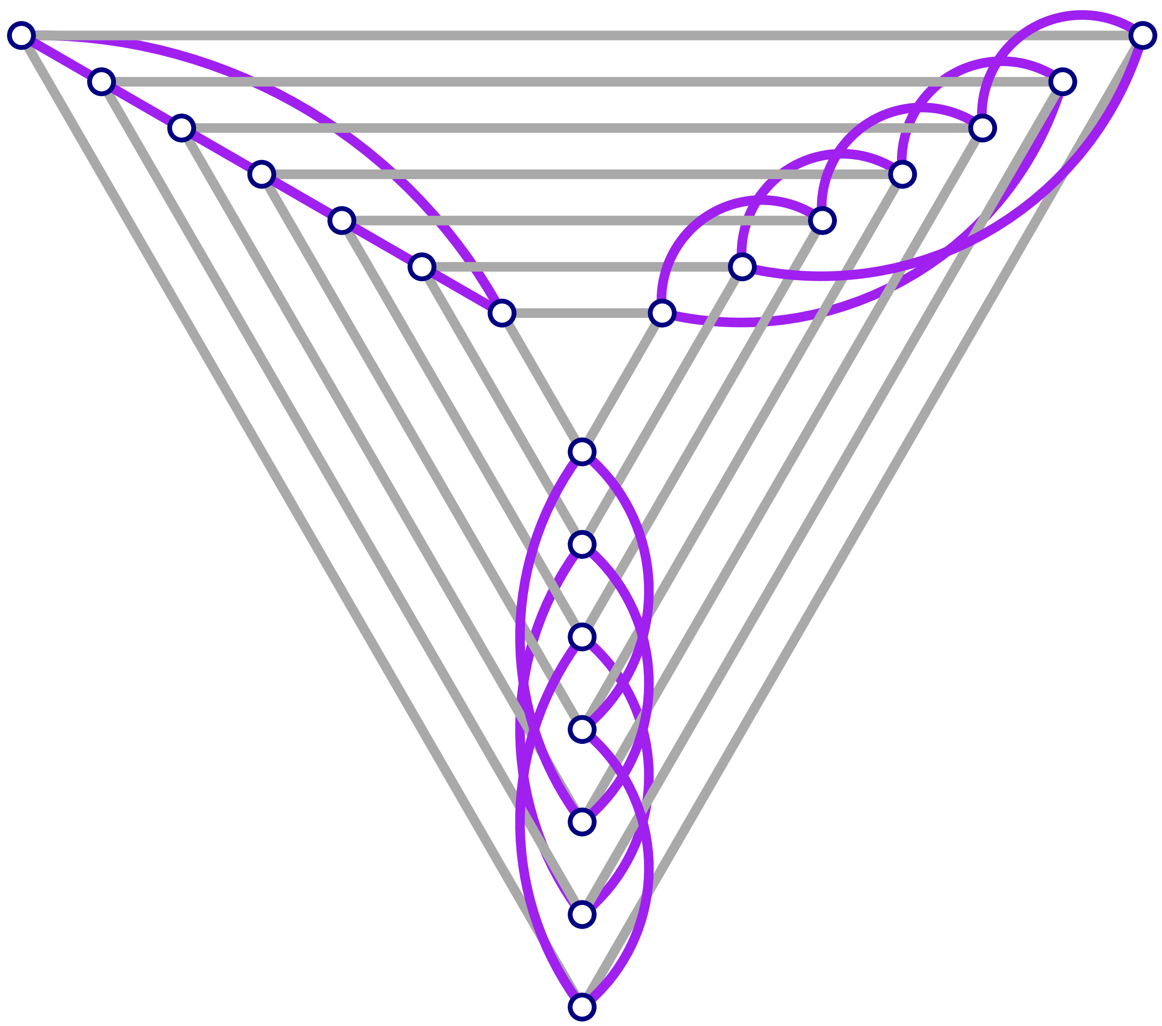
↖ abelian $\rtimes \mathbb{Z}_2$

More minimal Cayley graphs

$$\chi(\mathrm{Cay}(\mathbb{Z}_3 \rtimes \mathbb{Z}_7, \{(0,1),(1,0)\})) = 4$$



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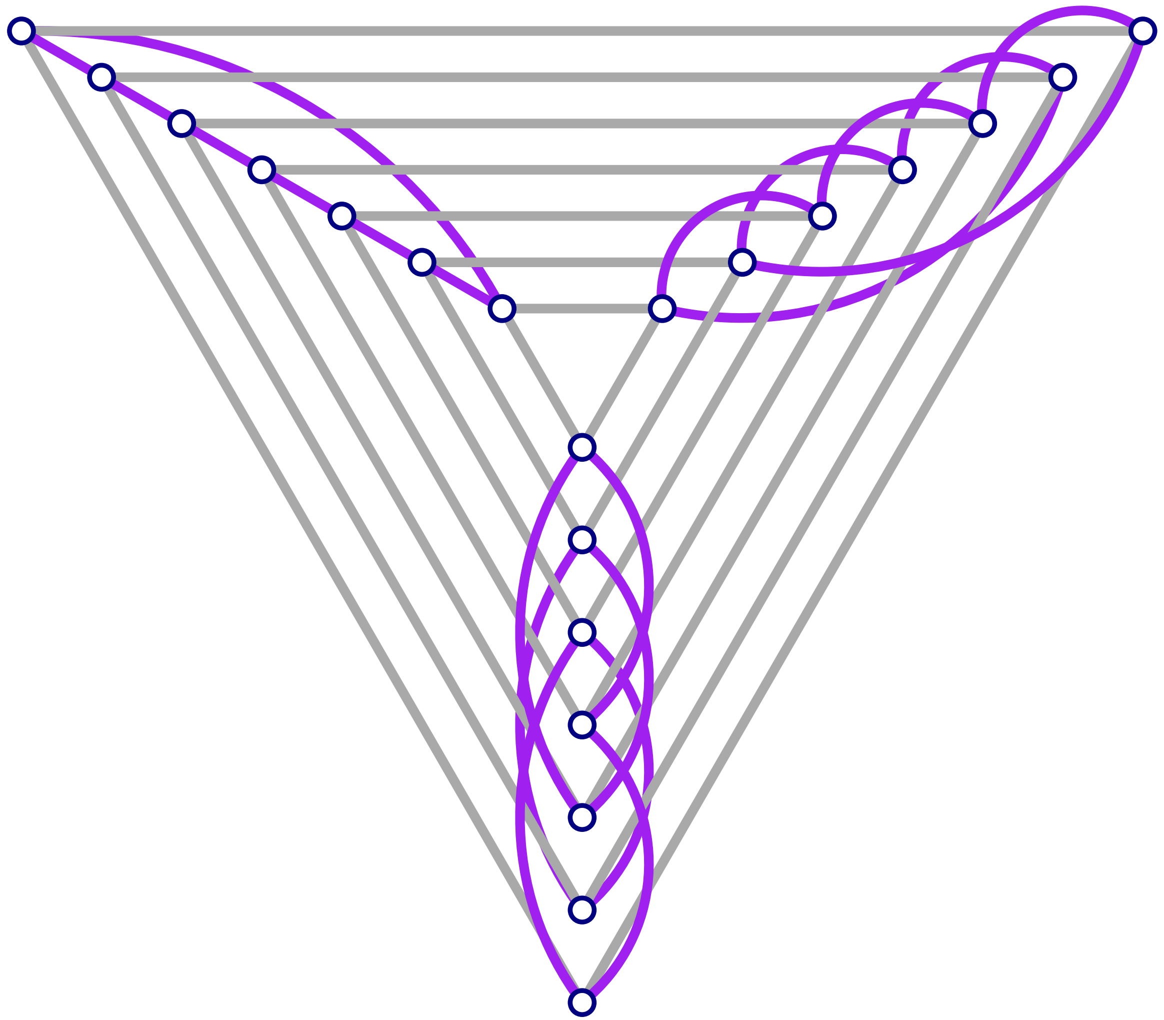

our **Thms** + **Frattini-Lemma** + **GAP** + **SageMath**

$\rightsquigarrow \chi \leq 4$ for all minimal Cayley on up to 223 vertices

Last slide

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$M \geq 4$

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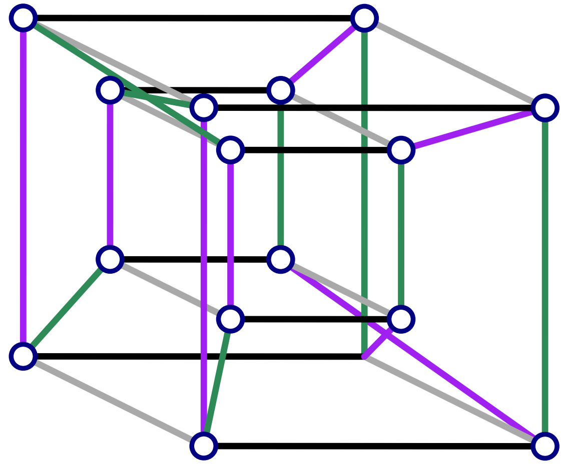
$\mathrm{Cay}(Q_{32}, C) = K_2 \boxtimes$ 

$\rightsquigarrow M \geq 7$

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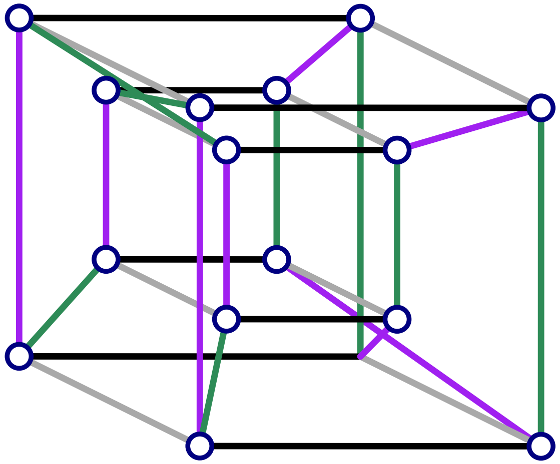
$\mathrm{Cay}(Q_{32}, C) = K_2 \boxtimes$ 

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How about semiminimal Cayley graphs of abelian groups?

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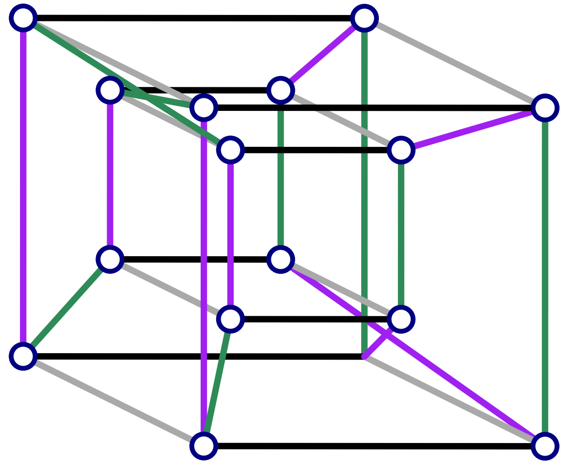
$\mathrm{Cay}(Q_{32}, C) = K_2 \boxtimes$ 

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Conj[Babai '96]: $\forall \varepsilon > 0 \exists$ minimal Cayley graph Γ such that $\alpha(\Gamma) \leq \varepsilon |\Gamma|$.

Last slide

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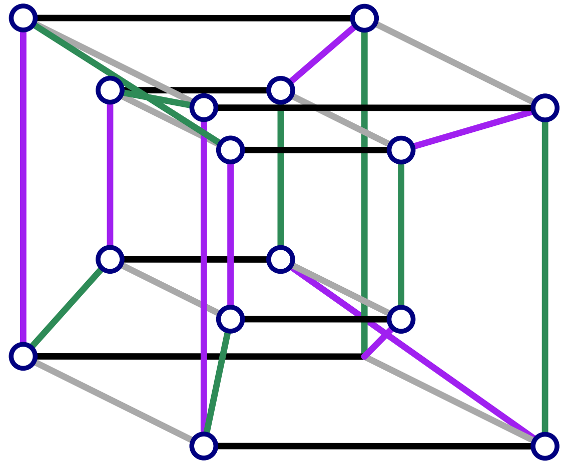
$\mathrm{Cay}(Q_{32}, C) = K_2 \boxtimes$ 

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How about graphs that can be partitioned into matching cuts?