Hilbert-Poincaré series in algebra and geometry

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Outline

The main object

Three viewpoints:

- $\diamond$ combinatorial
- $\diamond$ algebraic
- $\diamond$ geometrical $\to$ curve zeta functions $\to$ not today

Fragestellungen and exemplary results

Fragestellung 1. Commutative algebra

Fragestellung 2. Plane curve singularities

Main object

- Formal power series $\quad \leftrightarrow \quad Q \in \mathbb{Z}[\![t_1, \ldots, t_r]\!]$

$$r = 1: \qquad Q(t) = \sum_{n \in \mathbb{N}} a_n t^n$$

$$r > 1: \qquad Q(\underline{t}) = \sum_{\underline{n} \in \mathbb{N}^r} a_{\underline{n}} \underline{t}^{\underline{n}}, \qquad \underline{t}^{\underline{n}} := t_1^{n_1} \cdots t_r^{n_r}$$

Coefficients $a_{\underline{n}}$ can be endowed with an "interpretation".

- Historical milestones:

F. S. Macaulay (1913), A. Ostrowski (1922), . . ., I. Niven (1969)

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- Historical milestones:

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- Formal Laurent series $\longleftrightarrow$ $Q \in \mathbb{Z}[\![t_1, \ldots, t_r]\!][t_1^{-1}, \ldots, t_r^{-1}]$

Coefficients $a_{\underline{n}}$ with combinatorial meaning

Assume $S \subseteq \mathbb{N} := \{0, 1, \ldots\}$ is a numerical semigroup, i.e.

1. $0 \in S$
2. $m, n \in S \Longrightarrow m + n \in S$
3. There exists a conductor c i.e. $[c \longrightarrow] \in S$

For every $n \in \mathbb{N}$ define

$$a_n = \begin{cases} 1, & \text{if } n \in S; \\ 0, & \text{if } n \notin S. \end{cases}$$

The generating function of S is then

$$P_S(t) = \sum_{n \in \mathbb{N}} a_n t^n = \sum_{n \in S} t^n = \ldots = \frac{\Lambda_S(t)}{1 - t}$$

Note: Numerical semigroups occur in algebraic geometry e.g. as invariants of (plane) curve singularities, as we'll see.

Coin change problem: We have coins e.g. of 3 and 4. Set

$$a_n = \#(\text{ways of returning a value of } n \text{ with coins of 3 and 4}).$$

Consider $S = 3\mathbb{N} + 4\mathbb{N} = \langle 3, 4 \rangle$ so that

$$a_n = \begin{cases} \geq 1, & \text{if } n \in S; \\ 0, & \text{if } n \notin S. \end{cases}$$

This leads to the generating function

$$\sum_{n \in \mathbb{N}} a_n t^n = \ldots = \frac{1}{1 - t^3} \cdot \frac{1}{1 - t^4}$$

Observe that, for $S = \mathbb{N}$ we get $1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t}$.

This viewpoint can be thought in the language of commutative algebra.

Coefficients a_n with algebraic meaning

Let $\mathbb{K}$ be a field. Consider the polynomial ring $R = \mathbb{K}[X, Y]$ graded by $\deg(X) = 1, \deg(Y) = 1$.

R can be decomposed by the grading as

$$R = \bigoplus_{n=0}^{\infty} R_n = \underbrace{\mathbb{K}}_{R_0} \oplus (\underbrace{\mathbb{K}X \oplus \mathbb{K}Y}_{R_1}) \oplus \cdots \oplus (\underbrace{\text{span of monomials of deg } i}_{R_i}) \oplus \cdots$$

Define $a_n = \dim_{\mathbb{K}} R_n$. The generating series (Hilbert series of R) is

$$H_R(t) = \sum_{n \in \mathbb{N}} a_n t^n = \ldots = \frac{1}{1-t} \cdot \frac{1}{1-t}.$$

If R is endowed with the grading $\deg(X) = 3, \deg(Y) = 4$, then

$$H_R(t) = \sum_{n \in \mathbb{N}} a_n t^n = \ldots = \frac{1}{1-t^3} \cdot \frac{1}{1-t^4}.$$

This can be set in a more generality.

Coefficients $a_{\underline{n}}$ with algebraic meaning

Let $R := k[X_1, \ldots, X_n]$ be a polynomial ring endowed with a grading, typically

- ⋄ standard-$\mathbb{Z}$-grading, i.e., $\deg X_i = 1$
- ⋄ nonstandard-$\mathbb{Z}$-grading (positively graded)
- ⋄ $\mathbb{Z}^r$-grading (multigrading)

Let $0 \neq M = \bigoplus_\ell M_\ell$ be a finitely generated graded R-module, with Hilbert series

$$H_M(t) = \sum_{\ell \in \mathbb{Z}} (\dim_k M_\ell) t^\ell \in \mathbb{Z}[\![t]\!][t^{-1}]$$

Series without negative coefficients: nonnegative series.

Fragestellung 1: Commutative algebra

Let $M \neq 0$ be a finitely generated graded module over a graded polynomial ring $R = \mathbb{K}[X_1, X_2, \ldots, X_n]$.

Let H_M be the Hilbert series of M.

Note: arbitrary $\mathbb{N}^r$-grading, $r \geq 1$

Question 1.

(a) Since H_M is a rational function with "well understood" denominator (after Hilbert-Serre), which (Laurent) polynomials appear as potential numerators?

(b) Which series are Hilbert series of graded modules over polynomial rings?

Question 2. Which is the maximal depth of a finitely generated module M with Hilbert series H_M?

The answers may depend of the grading. For a $\mathbb{Z}^r$-grading we have

Theorem [—, Katthän, Uliczka]
A formal Laurent series H is the Hilbert series of a finitely generated R-module if and only if it can be written in the form

$$H = \sum_{I \subseteq [r]} \frac{Q_I}{\prod_{i \in I}(1 - \underline{t}^{\deg(X_i)})}$$

for Laurent polynomials $Q_I \in \mathbb{Z}[\![t_1^{\pm 1}, \ldots, t_r^{\pm 1}]\!]$ with nonnegative coefficients.

This is called a Hilbert decomposition of H.

◇ Useful for showing that a given Laurent series is a Hilbert series: construct a Hilbert decomposition.

◇ Difficult to show that a given series is *not* a Hilbert series.

◇ The coeff's of H behave as a polynomial function from some place on: Hilbert polynomial $\mathcal{P}(H)$ of H.

◇ For $I \subseteq [r]$, set $\mathbb{N}^I := \{\sum_{i \in I} c_i e_i : c_i \in \mathbb{N}\}$.

◇ Def: Let $H = \sum_{\underline{a}} \in \mathbb{Z}^r h_{\underline{a}} \underline{t}^{\underline{a}} \in \mathbb{Z}[\![\underline{t}]\!][\underline{t}^{-1}]$. For $I \subseteq [r]$ and $\underline{u} \in \mathbb{Z}^r$ we define the restriction of H to $\underline{u} + \mathbb{N}^I$ as

$$H\,|_{I,\underline{u}} := \sum_{\underline{a} \in \mathbb{N}^I} h_{\underline{u}+\underline{a}} \underline{t}^{\underline{a}} \in \mathbb{Z}[\![t_i]\!][t_i^{-1}]_{i \in I}.$$

◇ Def: Let $p \in \mathbb{Z}[W_1, \ldots, W_r]$ be a polynomial.

→ A monomial $\underline{W}^{\underline{b}}$ of p is said to be extremal if it doesn't divide any other monomial.

→ We say that p has positive extremal coefficients if the coefficient of every extremal monomial of p is positive.

Let m_i be the number of variables of degree e_i in R.

Theorem [—, Katthän, Uliczka]

For $H \in \mathbb{Z}[\![\underline{t}]\!][\underline{t}^{-1}]$, TFAE:

1. There exists a finit. generated graded R-module M with $H_M = H$.
2. H satisfies:
   (a) $H \cdot \prod_{i=1}^{r}(1 - t_i)^{m_i}$ is a polynomial;
   (b) for every $\underline{u} \in \mathbb{Z}^r$ and every $I \subseteq [r]$, the Hilbert polynomial $\mathcal{P}\Big(H\,|_{I,\underline{u}}\Big)$ has positive extremal coefficients.

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Example. Let $n = 2$, $m_1 = m_2 = 3$. Consider the series

$$H = \sum_{i \geq 0} \sum_{j \geq 0} (i-j)^2 t_1^i t_2^j = \frac{t_1 t_2^2 + t_1^2 t_2 + t_1^2 - 6 t_1 t_2 + t_2^2 + t_1 + t_2}{(1-t_1)^3 (1-t_2)^3}$$

Look at $\mathcal{P}(H) = (i-j)^2 = i^2 - 2ij + j^2$.

Let $\mathbb{K}[X, Y]$ be $\mathbb{N}$-graded by $\deg(X) = \alpha, \deg(X) = \beta$ coprime.

Let M be a f.g. graded R-module with Hilbert series H_M. Define

$\mathrm{Hdepth}(H_M) = \sup\left\{\mathrm{depth}(N) : N \text{ f.g. gr. } R\text{-module with } H_N = H_M\right\}$.

Theorem [—, Uliczka]

Write $H_M(t) = \sum_{n \in \mathbb{Z}} h_n t^n$. TFAE:

1. $\mathrm{Hdepth}(H_M) > 0$
2. For any $n \in \mathbb{Z}$ we have

$$\sum_{i \in I} h_{n+i} \leq \sum_{j \in J} h_{n+j}$$

   where

   - I is given by all minimal generator systems of "semimodules" of $\langle \alpha, \beta \rangle$ containing 0
   - J consists of the minimal generators of the corresponding syzygy semimodule of I

Fragestellung 2: Plane curve singularities

Consider a plane curve singularity C:$\{f = 0\}$ given by an analytically irreducible series $f := f(x, y)$.

We parametrize f by a map $\mathbb{C} \to \mathbb{C}^2$ with

$$\tau \mapsto (x(\tau), y(\tau)),$$

where $x(\tau), y(\tau) \in \mathbb{C}[\![\tau]\!]$ and $x(0) = y(0) = 0$. The set

$$S_C = \left\{\operatorname{ord}_\tau\big(g(x(\tau), y(\tau))\big) : g \in \mathbb{C}[\![x, y]\!],\ g \nmid f\right\}$$

is the value semigroup associated to C.

Independence of parametrization. Write

$$\nu(g) := \operatorname{ord}_\tau(g(x(\tau), y(\tau))).$$

For $n \in \mathbb{N}$, define $J(n) := \{g \in \mathbb{C}[\![x, y]\!]/(f) : \nu(g) \geq n\}$.

For any $n \in \mathbb{N}$ we have

$$J(n) \supseteq J(n+1) \quad \text{for any} \quad n \in \mathbb{N}$$
$$n \in S_C \iff \dim_{\mathbb{C}} \Big(J(n)/J(n+1) \Big) = 1.$$

If we associate a generating function to the filtration given by the ideals $J(n)$ by doing

$$a_n = \dim_{\mathbb{C}} \Big(J(n)/J(n+1) \Big),$$

we get the Poincaré series associated to C as $P_C(t) = \sum_{n \in \mathbb{N}} a_n t^n$.

This idea goes back to Campillo, Delgado, Kiyek ('94).

Observe: $P_C(t)$ coincides with $P_S(t)$.

Idea: P_C is a way of " counting" or "measuring" the elements g in the local ring of the singularity with fixed value $v(g) = n$.

If $f = f_1 \cdots f_r$ is reducible and reduced, then every branch $C_i : \{f_i = 0\}$ yields a valuation ν_i, and for $C = \cup_{i=1}^r C_i$ we define

$$S_C := \left\{ \underline{\nu}(g) := \big(\nu_1(g), \ldots, \nu_r(g)\big) \in \mathbb{Z}^r : g \in \mathbb{C}[\![x,y]\!], g \nmid f \right\}.$$

Which kind of subsemigroup of $(\mathbb{N}^r, +)$ is this $S = S_C$? Delgado:

(P1) $\underline{0} \in S$.

(P2) For $\underline{m} = (m_1, \ldots, m_r), \underline{n} = (n_1, \ldots, n_r) \in S$,

$$\inf(\underline{m}, \underline{n}) = (\min(m_1, n_1), \ldots, \min(m_r, n_r)) \in S.$$

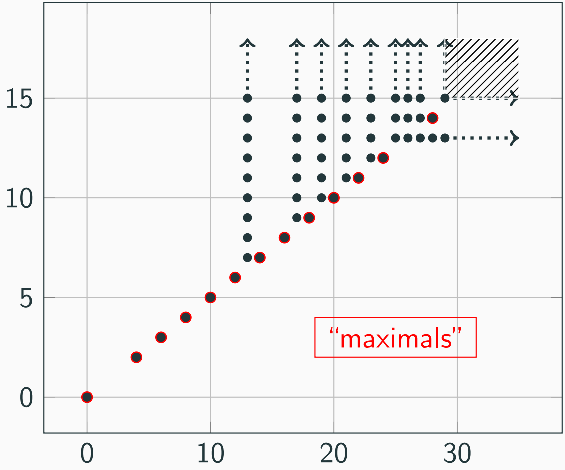
(P3) For $\underline{m}, \underline{n} \in S$ s.th. $\exists\ i_0$ with $m_{i_0} = n_{i_0}$, then $\exists\ \underline{z} \in S$ satisfying

$$z_k \geq \min(m_k, n_k) \text{ for all } k;$$
$$z_\ell = \min(m_\ell, n_\ell) \text{ if } m_\ell \neq n_\ell, \text{ and}$$
$$z_{i_0} > m_{i_0} = n_{i_0}.$$

(P4) There exists the minimum of the set $S \setminus \{\underline{0}\}$.

(P5) S has a conductor $c := \min\{z \in S : z + \mathbb{N}^r \subseteq S\}$.

Example: $C : \{f(x,y) = (y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7)(y^2 - x^3)\}$.



Question. How to define and compute $P_C(t_1, \ldots, t_r)$ or $P_S(\underline{t})$?

Answer (Campillo, Delgado, Gusein-Zade):

For $\underline{v} \in \mathbb{Z}^r$, the ideals

$$J(\underline{v}) := \{g \in \mathbb{C}[\![x, y]\!]/(f) : \underline{\nu}(g) \geq \underline{v}\},$$

define a multi-index filtration, since

$$J(\underline{v}) \supseteq J(\underline{w}) \quad \text{if and only if} \quad \underline{v} \leq \underline{w}$$

for the usual partial ordering "$\leq$" in $\mathbb{Z}^r$.

Notation: $\underline{1} = (1, 1, \ldots, 1)$, $e_i = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)$, $\underline{0} = (0, 0, \ldots, 0)$

A way to count dimensions $c(\underline{v}) := J(\underline{v})/J(\underline{v} + \underline{1})$:

$$L(\underline{t}) = \sum_{\underline{v} \in \mathbb{Z}^r} c(\underline{v}) \cdot \underline{t}^{\underline{v}} \in \mathbb{Z}[\![t_1^{\pm 1}, \ldots, t_r^{\pm 1}]\!].$$

Consider the power series in $\underline{t}$

$$L(\underline{t})\prod_{i=1}^{r}(t_i - 1) =: P'(\underline{t}) = \sum_{\underline{v}\in\mathbb{Z}^r} p'(\underline{v})\underline{t}^{\underline{v}}$$

with $e_J = \sum_{j\in J} e_j$ for $J \subseteq [r]$ and

$$\begin{aligned} p'(\underline{v}) &= (-1)^r \sum_{j=0}^{r}(-1)^j \sum_{1\le i_1<\cdots<i_j\le r} c(\underline{v} - e_{i_1} - \ldots - e_{i_r}) \\ &= (-1)^r \sum_{J\subseteq[r]} (-1)^{|J|} c(\underline{v} - e_J) \end{aligned}$$

Set $i \in [r]$ and $c(\underline{v}; i) := \dim J(\underline{v})/J(\underline{v} + e_i)$, then

$$c(\underline{v}; i) = 1 \iff \exists \underline{w} \in S \text{ with } w_i = v_i,\ w_j \geq v_j \ \ \forall j \in [r]$$

Fix $\underline{v} \in \mathbb{Z}^r$, $i \in [r]$. Define

$$p_i(\underline{v}) = (-1)^{r-1} \sum_{J \subseteq [r] \setminus \{i\}} (-1)^{|J|} c(\underline{v} + \underline{1} - e_i - e_J; i)$$
$$= (-1)^{r} \sum_{i \in L \subseteq [r]} (-1)^{|L|} c(\underline{v} + \underline{1} - e_L; i).$$

and

$$P_i(\underline{t}) = \sum_{\underline{v} \in \mathbb{Z}^r} p_i(\underline{v}) \underline{t}^{\underline{v}}.$$

Theorem

We have $p'(\underline{v}) = -p_i(\underline{v}) + p_i(\underline{v} - \underline{1})$ and

$$(t_1 t_2 \cdots t_r - 1) P_i(\underline{t}) = P'(\underline{t})$$

Hence $P_i(\underline{t})$ doesn't depend on i: we write $P(\underline{t}) = \sum_{\underline{v} \in \mathbb{Z}^r} p(\underline{v}) \underline{t}^{\underline{v}}$ and say *Poincaré series* of C. If $r > 1$, then $P(\underline{t})$ is a polynomial.

$P(\underline{t})$ depends only on S. Moreover

Theorem

(1) If $\underline{v} \notin S$, then $p(\underline{v}) = 0$.

(2) If $\underline{v} \in S$ is not maximal, then $p(\underline{v}) = 0$.

(3) If $\underline{v} \in S$ is absolute maximal, then $p(\underline{v}) = 1$.

(4) If $\underline{v} \in S$ is relative maximal, then $p(\underline{v}) = (-1)^r$.

Proof

- Fix r and consider any $\{i_1, \ldots, i_{r-1}\} \subseteq [r-1]$, then

$$0 \leq c(\underline{v} + \underline{1} - e_r; r) \leq c(\underline{v} + \underline{1} - e_r - e_1; r) \leq \cdots \leq c(\underline{v}; r) \leq 1.$$

• If $c(\underline{v} + \underline{1} - e_r; r) = 1$, all the terms in the sum below are 1:

$$p_r(\underline{v}) = (-1)^{r-1} \sum_{j=0}^{r-1} (-1)^j \sum_{1 \leq i_1 < \cdots < i_j \leq r-1} c(\underline{v} + \underline{1} - e_{i_1} - \cdots - e_{i_j}; r),$$

therefore

$$p_r(\underline{v}) = (-1)^{r-1} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} = (-1)^{r-1}(1-1)^{r-1} = 0.$$

• If $c(\underline{v}; r) = 0$, all terms involved in the above expression are 0, hence $p_r(\underline{v}) = p(\underline{v}) = 0$.

(1) If $\underline{v} \notin S$, then $\exists\ i \in [r]$ such that $c(\underline{v}; i) = 0$, hence $p_i(\underline{v}) = p(\underline{v}) = 0$.

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(2) If $\underline{v} \in S$, but not maximal, then $\exists\ i \in [r]$ such that $c(\underline{v} + \underline{1} - e_i; i) = 1$; therefore $p(\underline{v}) = p_i(\underline{v}) = 0$.

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(3) If $\underline{v} \in S$ absolute maximal, then $c(\underline{v}; r) = 1$ and, for every $J \subseteq [r] \setminus \{r\}$, we have $c(\underline{v} + \underline{1} - e_r - e_J; r) = 0$. This implies

$$p(\underline{v}) = (-1)^{r-1}(-1)^{r-1} = 1.$$

(4) If $\underline{v} \in S$ relative maximal, then $c(\underline{v} + \underline{1} - e_r; r) = 0$ and, for every $\emptyset \neq J \subseteq [r-1]$, we have $c(\underline{v} + \underline{1} - e_r - e_J; r) = 1$; hence

$$p(\underline{v}) = (-1)^{r-1} \sum_{j=1}^{r-1} (-1)^j \binom{r-1}{j} = (-1)^r.$$

The issue lies on those $\underline{v} \in S$ neither relative nor absolute maximal *but* maximal.

(!) $P(\underline{t})$ is a polynomial for $r > 1$.

If $r = 2$, then "relative maximal = absolute maximal = maximal" for S, and

$$P(t_1, t_2) = \sum_{(v_1, v_2) \text{ abs.max.}} t_1^{v_1} t_2^{v_2}.$$

If $r = 3$, then there are only absolute and relative maximal elements of S so that

$$P(t_1, t_2, t_3) = \sum_{(v_1, v_2, v_3) \text{ abs.max.}} t_1^{v_1} t_2^{v_2} t_3^{v_3} - \sum_{(v_1, v_2, v_3) \text{ rel.max.}} t_1^{v_1} t_2^{v_2} t_3^{v_3}.$$

Example

For $C : \{f(x,y) = (y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7)(y^2 - x^3)\}$, we have

$$
\begin{aligned}
P(t_1, t_2) = 1 &+ t_1^{4} t_2^{2} + t_1^{6} t_2^{3} + t_1^{8} t_2^{4} + t_1^{10} t_2^{5} + t_1^{12} t_2^{6} + t_1^{14} t_2^{7} + t_1^{16} t_2^{8} \\
&+ t_1^{18} t_2^{9} + t_1^{20} t_2^{10} + t_1^{22} t_2^{11} + t_1^{24} t_2^{12} + t_1^{28} t_2^{14} \\
&= \frac{(t_1^{12} t_2^{6} - 1)(t_1^{26} t_2^{13} - 1)}{(t_1^{4} t_2^{2} - 1)(t_1^{6} t_2^{3} - 1)}
\end{aligned}
$$